Estimation of Morisita Overlapping Measure λ for Two Weibull Distributions

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Abstract: The Morisita's measure λ is an overlapping coefficient between two probability density functions. It is defined as the similarity or agreement between two distributions. The parametric method to estimate this coefficient assumes that the underlying distributions for the two samples at hand are known with unknown parameters. Weibull distribution is one of the important statistical distributions because it is flexible distribution and it is used by scientist in different area of sciences. In this paper, new estimators for the Morisita measure λ are proposed under the parametric Weibull distributions. The derivation of the proposed estimators is based on a new technique that relies entirely on writing the formula of λ as expectation for some functions and then estimating the resulting expectation instead of estimating the exact integral of the formula of λ. In contrast to most studies in the literature, which assumed that the shape parameters of Weibulll distributions are equal, this paper proposes a new technique that enables us to derive new estimators for λ without putting any assumptions about the parameters of Weibull distributions. The performance of the resulting estimators are investigated and compared with the nonparametric kernel estimator and some existing estimators by performing an extensive simulation study. The numerical results showed that the performances of the proposed estimators are better than that of the kernel estimator for almost all considered cases.

Keywords: Morisita Measure; Maximum Likelihood Method; Parametric Method; Weibull Distribution; Relative Bias, Relative Root Mean Square Error and Efficiency

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1 Introduction

To determine the similarity between two populations in statistics, we use similarity measures. But if we want to make comparative inferences about two populations, we will use measures of dissimilarity. In fact, when we discuss this kind of measures, we should pay attention to the overlapping coefficients (OVL). Thus OVL are measures of agreement or similarity between probability distributions. For several important areas in which overlapping measures can be applied see [1] and [5]. The estimation of these OVL can be accomplished parametrically (see, [9, 11-13, 18]) or non-parametrically (see, [14-16]). Morisita measure λ is one of these measures, which is defined as follows:

Let $f_X(x)$ and $f_Y(y)$ be two probability density functions for the two independent continuous random variables X and Y respectively, the Morisita's meaure is defined by [20],

$$\lambda = \frac{2\int f_1(x) f_2(x) dx}{\int [f_1(x)]^2\, dx + [f_2(x)]^2\, dx}$$

If the value of λ is 1 then $f_1(x) = f_2(x)$. If the value of $\lambda = 0$ then the supports of the two densities have no interior points in common.

Al-Saidy *et al.* [2] derived the formula of λ for two Weibull distributions (each with two parameters) having the same shape parameter but different scale parameters.

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The pdf of the Weibull model with scale parameter α and shape parameter β is,

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}}, x > 0, \alpha, \beta > 0.$$

which is denoted by $We(\beta,\alpha)$. The Weibull distribution is associated with a number of probability density functions. In particular, it interpolates between the exponential distribution ($\beta = 1$) and the Rayliegh distribution ($\beta = 2$ and $\alpha = \sqrt{2}\sigma$).

Now, let the two independent continuous random variables $X \sim We(\beta_1,\alpha_1)$ and $Y \sim We(\beta_2,\alpha_2)$. Assume that $\beta_1 = \beta_2 = \beta$ and let $K = \alpha_1/\alpha_2$ and $Q = (2\beta - 1)/\beta$, then the formula of the measure λ based on these two Weibull distributions was derived by [2], which is given as,

$$\lambda = \frac{2\int_0^{\infty} \mathrm{W}(\alpha_1,\beta)\mathrm{W}(\alpha_2,\beta)\mathrm{dx}}{\int_0^{\infty} W^2(\alpha_1,\beta)\mathrm{dx}+\int_0^{\infty}\mathrm{W}^2(\alpha_2,\beta)} = \frac{2^{\mathrm{Q}+1}\mathrm{K}^{\beta}}{(1+\mathrm{K})(1+\mathrm{K}^{\beta})^{\mathrm{Q}}}.$$

The OVL measure λ for two exponential distributions can be obtained by taking $\beta = 1$ in the Weibull distribution. The case of two exponential distributions was studied by [19] and [21], who also studied the effect of sampling scheme on OVL measures.

If $\alpha_1 = \alpha_2$ but $\beta_1 \neq \beta_2$ or if $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$ then λ cannot be computed in closed form and then an estimator for λ cannot be developed. Without assuming any restrictions on the parameters of Weibull distributions, this paper suggested a new technique to deal with the formula of λ, which in turn leads to new proposed estimators for λ.

2 New Technique

The Morisita Measureλ can be expressed as follows,

$$\lambda = \frac{2R_{XY}}{H_X+S_Y}$$

where

$$R_{XY} = \int_0^{\infty} f_X(x)f_Y(x)dx,$$

$$H_X = \int_0^{\infty}[f_X(x)]^2dx$$

and

$$S_Y = \int_0^{\infty}[f_Y(x)]^2dx$$

We propose to write each of the above quantities R_{XY}, H_X and S_Y as expected value of some function as the following,

$$Ef_X(Y) = \int_0^{\infty} f_X(y)f_Y(y)dy$$

$$= \int_0^{\infty} f_X(x)f_Y(x)dx = R_{XY}$$

Also,

$$Ef_Y(X) = \int_0^{\infty} f_Y(x)f_X(x)dx = R_{XY}$$

That is, $R_{XY} = Ef_X(Y) = Ef_Y(X) = \frac{1}{2}(Ef_X(Y) + Ef_Y(X))$. The other two quantities H_X and S_Y can be expressed as follows,

$$Ef_X(X) = \int_0^{\infty} f_X(x)f_X(x)dx$$

$$= \int_0^{\infty}[f_X(x)]^2dx = H_X$$

and

$$Ef_Y(Y) = \int_0^{\infty} f_Y(y)f_Y(y)dy$$

$$= \int_0^{\infty} f_Y(x)f_Y(x)dx$$

$$= \int_0^{\infty}[f_Y(x)]^2dx = S_Y.$$

Therefore, the Morisita Measure λ can be expressed as follows,

$$\lambda = \frac{2R_{XY}}{H_X+S_Y} = \frac{2Ef_X(Y)}{Ef_X(X)+Ef_Y(Y)} = \frac{2Ef_Y(X)}{Ef_X(X)+Ef_Y(Y)} = \frac{Ef_X(Y)+Ef_Y(X)}{Ef_X(X)+Ef_Y(Y)}.$$

3. Estimation of λ

To estimate λ based on two independent random samples $X_1, X_2, \dots, X_{n_1}$ and $Y_1, Y_2, \dots, Y_{n_2}$, we suggest to estimate each of the quantities R_{XY}, H_X and S_Y separately. The idea of estimating R_{XY}, H_X and S_Y depends on the well-known method called "method of moments", which can be found in any intermediate mathematical statistical book.

Since $H_X = Ef_X(X)$ is the mean of $f_X(X)$, we suggest to estimate it by the mean of $f_X(X_k), k = 1,2,\dots,n_1$. However, the Weibull distribution $f_X(X_k)$ is still depends on the two parameters α_1 and β_1. Let $f_X(X_k;\hat{\alpha}_1,\hat{\beta}_1)$ be the MLE of $f_X(X_k)$, then the estimator $\widehat{H}_X$ of H_X is suggested to be,

$$\widehat{H}_X = \frac{1}{n_1}\sum_{k=1}^{n_1} f_X(X_k;\hat{\alpha}_1,\hat{\beta}_1).$$

By the same way, we can estimate $S_Y = Ef_Y(Y)$ by the mean of $f_Y(Y_k), k = 1,2,\dots,n_2$. Let $f_Y(Y_k;\hat{\alpha}_2,\hat{\beta}_2)$ be the MLE of $f_Y(Y_k)$, then the suggested estimator of S_Y is,

$$\hat{S}_Y = \frac{1}{n_2}\sum_{k=1}^{n_2} f_Y\left(Y_k; \hat{\alpha}_2, \hat{\beta}_2\right).$$

Also, $R_{XY} = Ef_X(Y) = Ef_Y(X) = \frac{1}{2}\left(Ef_X(Y) + Ef_Y(X)\right)$ can be estimated by the mean of $f_X(Y_k), k = 1,2,\dots,n_2$ or the mean of $f_Y(X_k), k = 1,2,\dots,n_1$ or the average of their means. That is, the estimator $\hat{R}_{XY}$ of R_{XY} is suggested to be,

$$\hat{R}_{XY} = \frac{1}{n_2}\sum_{k=1}^{n_2} f_X\left(Y_k; \hat{\alpha}_1, \hat{\beta}_1\right)$$

or

$$\hat{R}_{XY} = \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y\left(X_k; \hat{\alpha}_2, \hat{\beta}_2\right)$$

or

$$\hat{R}_{XY} = \frac{1}{2}\left(\frac{1}{n_2}\sum_{k=1}^{n_2} f_X\left(Y_k; \hat{\alpha}_1, \hat{\beta}_1\right) + \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y\left(X_k; \hat{\alpha}_2, \hat{\beta}_2\right)\right).$$

The preliminary simulation results indicate that the last estimator of R_{XY} is more stable than the other previous two estimators. Therefore, the simulation study in Section (5) considered only the last estimator. Based on the previous derivations, the proposed estimator Morisita Measure λ is,

$$\hat{\lambda} = \frac{\hat{R}_{XY}}{\hat{H}_X + \hat{S}_Y} = \frac{\frac{1}{n_2}\sum_{k=1}^{n_2} f_X(Y_k;\hat{\alpha}_1,\hat{\beta}_1) + \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y(X_k;\hat{\alpha}_2,\hat{\beta}_2)}{\frac{1}{n_1}\sum_{k=1}^{n_1} f_X(X_k;\hat{\alpha}_1,\hat{\beta}_1) + \frac{1}{n_2}\sum_{k=1}^{n_2} f_Y(Y_k;\hat{\alpha}_2,\hat{\beta}_2)}$$

Finally, we state the following important note.

Note: The MLEs of the different parameters of Weibull distribution were obtained for the three cases: a) two shape parameters are equal b) two scale parameters are equal and c) no restrictions on the scale and shape parameters. For the last case, the estimator of λ is exactly as given above. For Case (a), $f_X(.)$ and $f_Y(.)$ that arise in formula of $\hat{\lambda}$ should be $f_X(.) = f_X\left(.; \hat{\alpha}_1, \hat{\beta}\right)$ and $f_Y = f_Y\left(.; \hat{\alpha}_2, \hat{\beta}\right)$ and for Case (b) they should be $f_X(.) = f_X\left(.; \hat{\alpha}, \hat{\beta}_1\right)$ and $f_Y = f_Y\left(.; \hat{\alpha}, \hat{\beta}_2\right)$.

4 Estimation of Weibull Distributions Parameters

As stated in the previous section, to estimate the Morisita measure λ we need to estimate the parameters of the two Weibull distributions. We suggest using the maximum likelihood method, which has been briefly described in this section.

Let $X_1, X_2, \dots, X_{n_1}$ be a random sample of size n_1 from $W(x; \alpha_1, \beta_1)$ and let $Y_1, Y_2, \dots, Y_{n_2}$ be another random sample of size n_2 from $W(y; \alpha_2, \beta_2)$, where the two samples are independent. The likelihood function of $X_1, X_2, \dots, X_{n_1}$ and $Y_1, Y_2, \dots, Y_{n_2}$ is,

$$L(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{{\beta_1}^{n_1}}{{\alpha_1}^{n_1\beta_1}} \frac{{\beta_2}^{n_2}}{{\alpha_2}^{n_2\beta_2}} \left(\prod_{i=1}^{n_1} x_i\right)^{\beta_1 - 1} \left(\prod_{i=1}^{n_2} y_i\right)^{\beta_2 - 1} e^{-\frac{1}{\alpha_1^{\beta_1}}\sum_{i=1}^{n_1} x_i^{\beta_1} - \frac{1}{\alpha_2^{\beta_2}}\sum_{i=1}^{n_2} y_i^{\beta_2}}$$

In the following subsections, the maximum likelihood estimators (MLEs) of the different Weibull distribution parameters are obtained by considering the three cases:

a) The two scale parameters are equal
b) The two shape parameters are equal and
c) There is no restriction about the scale and shape parameters.

4.1 Two Scale Parameters Are Equal

Suppose that the two scale parameters are equal, i.e. $\alpha_1 = \alpha_2 = (\alpha$ say), then the likelihood function becomes (see also, [10]),

$$L(\alpha, \beta_1, \beta_2) = \frac{{\beta_1}^{n_1}{\beta_2}^{n_2}}{\alpha^{n_1\beta_1 + n_2\beta_2}} \left(\prod_{i=1}^{n_1} x_i\right)^{\beta_1 - 1} \left(\prod_{i=1}^{n_2} y_i\right)^{\beta_2 - 1} e^{-\frac{1}{\alpha^{\beta_1}}\sum_{i=1}^{n_1} x_i^{\beta_1} - \frac{1}{\alpha^{\beta_2}}\sum_{i=1}^{n_2} y_i^{\beta_2}}$$

The ML estimators of α, β_1 and β_2 can be obtained by solving the following three equations numerically with respect to α, β_1 and β_2,

$$\frac{\partial lnL(\alpha,\beta_1,\beta_2)}{\partial\alpha} = -\frac{n_1\beta_1 + n_2\beta_2}{\alpha} + \beta_1\frac{\sum_{i=1}^{n_1} {x_i}^{\beta}}{\alpha^{\beta_1+1}} + \beta_2\frac{\sum_{i=1}^{n_2} {y_i}^{\beta_2}}{\alpha^{\beta_2+1}} = 0$$

$$\frac{\partial lnL(\alpha,\beta_1,\beta_2)}{\partial\beta_1} = \frac{n_1}{\beta_1} - n_1\, ln\alpha + \sum_{i=1}^{n_1} ln\, x_i - \sum_{i=1}^{n_1}\left[\left(\frac{x_i}{\alpha}\right)^{\beta_1} ln\left(\frac{x_i}{\alpha}\right)\right] = 0$$

$$\frac{\partial lnL(\alpha,\beta_1,\beta_2)}{\partial\beta_2}=\frac{n_2}{\beta_2}-n_2\,ln\alpha+\sum_{i=1}^{n_2}ln\,y_i-\sum_{i=1}^{n_2}\left[\left(\frac{y_i}{\alpha}\right)^{\beta_2}ln\left(\frac{y_i}{\alpha}\right)\right]=0.$$

4.2 Two Shape Parameters Are Equal

Suppose that the two shape parameters are equal, i.e. $\beta_1=\beta_2=(\beta$ say), then the likelihood function becomes,

$$L(\alpha_1,\alpha_2,\beta)=\frac{\beta^{n_1+n_2}}{\alpha_1{}^{n_1\beta}\alpha_2{}^{n_2\beta}}\left(\prod_{i=1}^{n_1}x_i\prod_{i=1}^{n_2}y_i\right)^{\beta-1}e^{-\frac{1}{\alpha_1{}^{\beta}}\sum_{i=1}^{n_1}x_i^{\beta}-\frac{1}{\alpha_2{}^{\beta}}\sum_{i=1}^{n_2}y_i^{\beta}},$$

The ML estimators of α_1,α_2 and β are obtained by solving the following three equations numerically,

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta)}{\partial\alpha_1}=\frac{-n_1\beta}{\alpha_1}+\frac{\beta\sum_{i=1}^{n_1}x_i^{\beta}}{\alpha_1{}^{\beta+1}}=0$$

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta)}{\partial\alpha_2}=\frac{-n_2\beta}{\alpha_2}+\frac{\beta\sum_{i=1}^{n_2}y_i^{\beta}}{\alpha_2{}^{\beta+1}}=0$$

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta)}{\partial\beta}=\frac{n_1+n_2}{\beta}-\left(n_1ln\alpha_1+n_2ln\alpha_2\right)+\left(\sum_{i=1}^{n_1}lnx_i+\sum_{i=1}^{n_2}lny_i\right)$$

$$-\sum_{i=1}^{n_1}\left[\left(\frac{x_i}{\alpha_1}\right)^{\beta}ln\left(\frac{x_i}{\alpha_1}\right)\right]-\sum_{i=1}^{n_2}\left[\left(\frac{y_i}{\alpha_2}\right)^{\beta}ln\left(\frac{y_i}{\alpha_2}\right)\right]=0$$

4.3 Scale and Shape Parameters Are Not Necessarily Equal

Without using any restrictions about the Weibull distributions parameters, the likelihood function $L(\alpha_1,\alpha_2,\beta_1,\beta_2)$ is as given above. Therefore, the ML estimators of $\alpha_1,\alpha_2,\beta_1$ and β_2 are obtained by solving the following equations simultaneously with respect to $\alpha_1,\alpha_2,\beta_1$ and β_2.

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta_1,\beta_2)}{\partial\alpha_1}=\frac{-n_1\beta_1}{\alpha_1}+\frac{\beta_1\sum_{i=1}^{n_1}x_i^{\beta_1}}{\alpha_1{}^{\beta_1+1}}=0$$

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta_1,\beta_2)}{\partial\alpha_2}=\frac{-n_2\beta_2}{\alpha_2}+\frac{\beta_2\sum_{i=1}^{n_2}y_i^{\beta_2}}{\alpha_2{}^{\beta_2+1}}=0$$

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta_1,\beta_2)}{\partial\beta_1}=\frac{n_1}{\beta_1}-n_1\,ln\alpha_1+\sum_{i=1}^{n_1}ln\,x_i-\sum_{i=1}^{n_1}\left[\left(\frac{x_i}{\alpha_1}\right)^{\beta_1}ln\left(\frac{x_i}{\alpha_1}\right)\right]=0$$

$$\frac{\partial lnL(\alpha_1,\alpha_2,\beta_1,\beta_2)}{\partial\beta_2}=\frac{n_2}{\beta_2}-n_2\,ln\alpha_2+\sum_{i=1}^{n_2}ln\,y_i-\sum_{i=1}^{n_2}\left[\left(\frac{y_i}{\alpha_2}\right)^{\beta_2}ln\left(\frac{y_i}{\alpha_2}\right)\right]=0$$

5 Simulation Study

A simulation study is conducted to compare the performances of the proposed estimators of the overlapping measures λ with the existing counterpart estimators that are developed in the literature. In particular, the nonparametric kernel estimator is the only general estimator that is available for this purpose. Some other estimators are developed under some conditions. It is important to note that the kernel method produces general estimators, which does not require any assumptions about the shape of the underlying sample distribution, (See [14]). To be consistent with the derivations of this paper, we considered the following three cases:

Case (1). A pair of Weibull distributions with the same scale parameters (i.e. $\alpha_1=\alpha_2$).

Case (2). A pair of Weibull distributions with the same shape parameters (i.e. $\beta_1=\beta_2$).

Case (3). A pair of Weibull distributions with different scale parameters and different shape parameters (i.e. $\alpha_1\neq\alpha_2$ and $\beta_1\neq\beta_2$).

We simulated $x_1,x_2,\dots,x_{n_1}$ from $f_X(x)=We(\alpha_1,\beta_1)$ and $y_1,y,\dots,y_{n_2}$ from $f_Y(y)=We(\alpha_2,\beta_2)$, where $f_X(x)$ and $f_Y(y)$ and the corresponding choosing parameters are given in Cases (1), (2) and (3). To study the behavior of each estimator for different sample sizes, (n_1, n_2) are taken to be (n_1,n_2) = (10,10), (20,30), (30,30), (50,50), (100,200).

The empirical results were calculated based on one thousand replication (R = 1000), which is achieved by using Mathematica, Version 7. For each estimator, we compute the Relative Bias (RB), Relative Root Mean Square Error (RRMSE) and Efficiency Root (EFR). They are defined as:

$$RB = \frac{\hat{E}(estimator) - exact}{exact},$$

$$RRMSE = \frac{\sqrt{\widehat{MSE}(estimator)}}{exact}$$

and

$$EFR = \sqrt{\frac{\widehat{MSE}(kernel)}{\widehat{MSE}(proposed\ estimator)}}.$$

For example, if $\hat{\lambda}$ is the estimator of λ and if $\hat{\lambda}_{(j)}$ is the value of $\hat{\lambda}$ computed based on a sample of iteration j, $j = 1, 2, \ldots, Reb = 1000$ then,

$$\hat{E}(\hat{\lambda}) = \sum_{j=1}^{R} \hat{\lambda}_{(j)} / \mathrm{Rep}$$

and

$$\widehat{MSE}(\hat{\lambda}) = \sum_{j=1}^{R} \left(\hat{\lambda}_{(j)} - E(\hat{\lambda})\right)^2 / Rep.$$

Which implies that the corresponding empirical measures are,

$$RB = \frac{\hat{E}(\hat{\lambda}) - \lambda}{\lambda},$$

$$RRMSE = \frac{\sqrt{\widehat{MSE}(\hat{\lambda})}}{\lambda}$$

and the EFR with respect to estimator " $kernel$ " is,

$$EFR = \sqrt{\frac{\widehat{MSE}(kernel)}{\widehat{MSE}(\hat{\lambda})}}.$$

For Case (1): The two proposed estimators of λ are,

$$\hat{\lambda}_{p1} = \frac{\frac{1}{n_2}\sum_{k=1}^{n_2} f_X(Y_k;\hat{\alpha}_1,\hat{\beta}_1) + \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y(X_k;\hat{\alpha}_2,\hat{\beta}_2)}{\frac{1}{n_1}\sum_{k=1}^{n_1} f_X(X_k;\hat{\alpha}_1,\hat{\beta}_1) + \frac{1}{n_2}\sum_{k=1}^{n_2} f_Y(Y_k;\hat{\alpha}_2,\hat{\beta}_2)}$$

and

$$\hat{\lambda}_{p2} = \frac{\frac{1}{n_2}\sum_{k=1}^{n_2} f_X(Y_k;\hat{\alpha},\hat{\beta}_1) + \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y(X_k;\hat{\alpha},\hat{\beta}_2)}{\frac{1}{n_1}\sum_{k=1}^{n_1} f_X(X_k;\hat{\alpha},\hat{\beta}_1) + \frac{1}{n_2}\sum_{k=1}^{n_2} f_Y(Y_k;\hat{\alpha},\hat{\beta}_2)}$$

For comparison purposes, the counterparts of nonparametric kernel estimators (See, [14]) have been also investigated. This estimator was coded by $\hat{\lambda}_k$ and its bandwidth parameter h is compute by using the formula,

$$h = 1.06\, S\, n^{-1/5}$$

where S is the usual standard deviation for the interested sample. Another ways to compute h are possible (see [17]), however, we found that the above choice to compute h gives acceptable results. Note that the formulas for kernel estimators for the previous different three cases remain unchanged.

Note: The above proposed estimator $\hat{\lambda}_{p1}$ is also used for Cases (2) and (3), while the proposed estimator $\hat{\lambda}_{p2}$ for Case (2) is given as follows,

$$\hat{\lambda}_{p2} = \frac{\frac{1}{n_2}\sum_{k=1}^{n_2} f_X(Y_k;\hat{\alpha}_1,\hat{\beta}) + \frac{1}{n_1}\sum_{k=1}^{n_1} f_Y(X_k;\hat{\alpha}_2,\hat{\beta})}{\frac{1}{n_1}\sum_{k=1}^{n_1} f_X(X_k;\hat{\alpha}_1,\hat{\beta}) + \frac{1}{n_2}\sum_{k=1}^{n_2} f_Y(Y_k;\hat{\alpha}_2,\hat{\beta})}.$$

Since the other two estimators $\hat{\lambda}_{p1}$ and $\hat{\lambda}_k$ can be applied to Case (2), they are also considered in this case.

For Case (3), only the proposed estimator $\hat{\lambda}_{p1}$ and the kernel estimators $\hat{\lambda}_k$ are considered and investigated.

6 Simulation Results

All computations and outputs of the simulation study are presented in Tables 1, 2 and 3. The entries of each table represents RB, RRMSE and EFR (with respect to kernel estimator $\hat{\lambda}_k$). The results of Table 1 depend on the data simulated from pair Weibull distributions with the equal scale parameters. Table 2 contains the results when the data are simulated from pair Weibull distribution with equal shape parameters, while the results of Table 3 depends on the data simulated from pair Weibull distributions with different scale and different shape parameters.

1. It is obvious that the $|RB|$s that are associated with the kernel estimators $\hat{\lambda}_k$ are large compared with other estimators, especially for small samples sizes. Most RBs values of the kernel estimates are negative, which indicates that -on the average- the kernel estimates underestimate the true value of the corresponding parameter. The underestimation issue seems to be a problem associated with the kernel estimates, even in other fields such as line transect sampling (see, [6-8]) and degradation method (see, [3] and [4]).

2. The values $|RB|$s of the different proposed estimates are much smaller than that of the kernel estimates for almost all considered cases. Actually, the RBs that associated with different suggested estimates seem to be very acceptable.

3. As the samples sizes increase the RRMSE of the different estimators decrease. This is a good sign for the consistency of the different estimators that considered in this study.

4. The values of $RRMSEs$ and consequently the values of EFRs for the proposed estimates of different cases indicate that they perform better than the kernel estimates.

5. The proposed estimators $\hat{\lambda}_{p1}$ perform very well even when the data are simulated from pair Weibull distributions with equal scales or with equal shape parameters. By taking into account that $\hat{\lambda}_{p1}$ is developed without any constraints on the pair Weibull distributions parameters, then we can recommend them as general estimators for λ.

7 Conclusion

Writing the formula of Morisita coefficient λ as expected value of some functions and then deriving new estimators of λ under a pair of Weibull distributions is the main purpose of this paper. The benefit of this technique is to estimate λ without placing any assumptions on the parameters of Weibull distributions. Based on the numerical results obtained by performing a simulation study, the results demonstrated clearly the effectiveness of the new technique and that the performance of the estimator resulting from the use of this technique is better than the performance of the nonparametric kernel estimator of λ that developed by Eidous and AL-Talafha [9]. This technique can also be used to estimate other overlapping coefficients mentioned in the literature, such as the Weitzmann and Matusita coefficients (see, [15] and [16]) and Pianka and Kullback-Leibler coefficients (see, [11]).

Table 1 The RB, RRMSE and EFR of the three estimators $\hat{\lambda}_k, \hat{\lambda}_{p_1}$ and $\hat{\lambda}_{p_2}$ when the data are simulated from pair Weibull distributions with equal scale parameters

(n_1, n_2)		$\hat{\lambda}_k$	$\hat{\lambda}_{p_1}$	$\hat{\lambda}_{p_2}$	$\hat{\lambda}_k$	$\hat{\lambda}_{p_1}$	$\hat{\lambda}_{p_2}$
	$\lambda_{exact} = 0.9710$, $(\beta_1, \beta_2) = (3, 4)$				$\lambda_{exact} = 0.8373$, $(\beta_1, \beta_2) = (3, 6.2)$		
(10, 10)	RB	-0.4595	-0.0912	-0.0143	-0.5866	-0.1430	-0.0593
	RRMSE	0.4875	0.1427	0.0833	0.6169	0.2389	0.1844
	EFR	1.0000	3.4158	5.8524	1.0000	2.5818	3.3450
(20, 30)	RB	-0.2582	-0.0453	-0.0100	-0.3945	0.0011	0.0612
	RRMSE	0.2833	0.0966	0.0715	0.4117	0.0867	0.1170
	EFR	1.0000	2.9324	3.9625	1.0000	4.7476	3.5180
(30, 30)	RB	-0.2583	-0.0183	0.0078	-0.2786	0.0137	0.0482
	RRMSE	0.2676	0.0428	0.0320	0.3037	0.0896	0.1020
	EFR	1.0000	6.2415	8.3622	1.0000	3.3894	2.9761
(50, 50)	RB	-0.1920	-0.0248	-0.0072	-0.2469	-0.0261	0.0073
	RRMSE	0.2058	0.0442	0.0353	0.2761	0.1004	0.0929
	EFR	1.0000	4.6485	5.8212	1.0000	2.7486	2.9722
(100, 200)	RB	-0.0971	-0.0035	0.0188	-0.0994	0.0026	0.0506
	RRMSE	0.1000	0.0190	0.0225	0.1168	0.0500	0.0704
	EFR	1.0000	0.1281	1.1751	1.0000	2.3356	1.6601
	$\lambda_{exact} = 0.6249$, $(\beta_1, \beta_2) = (3, 10.3)$				$\lambda_{exact} = 0.3701$, $(\beta_1, \beta_2) = (3, 20.4)$		
		-0.6313	-0.0712	0.0899	-0.8214	-0.0014	0.0345
(10, 10)	RB	0.6803	0.2469	0.2545	0.8370	0.4858	0.5068
	RRMSE	1.0000	2.7548	2.6731	1.0000	1.7229	1.6514
	EFR	-0.4694	-0.0817	0.0147	-0.5790	-0.0339	0.0618
(20, 30)	RB	0.4948	0.1518	0.1632	0.6138	0.2618	0.3373
	RRMSE	1.0000	3.2584	3.0309	1.0000	2.3444	1.8198
	EFR	-0.4339	-0.0448	0.0017	-0.5624	0.0081	0.1420
(30, 30)	RB	0.4613	0.1895	0.2272	0.5921	0.2451	0.3053
	RRMSE	1.0000	2.4339	2.0306	1.0000	2.4158	1.9394
	EFR	-0.3815	-0.0420	0.0290	-0.5141	-0.0517	0.0467
(50, 50)	RB	0.4017	0.1189	0.1084	0.5384	0.1842	0.2096
	RRMSE	1.0000	3.3777	3.7040	1.0000	2.9223	2.5686

(n_1, n_2)		$\hat{\lambda}_k$	$\hat{\lambda}_{p_1}$	$\hat{\lambda}_{p_2}$	$\hat{\lambda}_k$	$\hat{\lambda}_{p_1}$	$\hat{\lambda}_{p_2}$
	EFR	-0.1134	0.0119	0.0842	-0.2042	0.0231	0.0780
(100, 200)	RB	0.1506	0.0858	0.0921	0.2388	0.0877	0.1180
	RRMSE	1.0000	1.7538	1.5751	1.0000	2.7216	2.0227
	EFR	-0.6313	-0.0712	0.0899	-0.8214	-0.0014	0.0345

Table 2 The RB, RRMSE and EFR of the three estimators $\hat{\lambda}_k, \hat{\lambda}_{p_1}$ and $\hat{\lambda}_{p_2}$ when the data are simulated from pair Weibull distributions with equal shape parameters

(n_1, n_2)		$\hat{\lambda}_k$	$\hat{\lambda}_{p_1}$	$\hat{\lambda}_{p_2}$	$\hat{\lambda}_k$	$\hat{\lambda}_{p_1}$	$\hat{\lambda}_{p_2}$
	$\lambda_{\text{exact}} = 0.9363$, $(\alpha_1, \alpha_2) = (1, 1.2)$				$\lambda_{\text{exact}} = 0.7324$, $(\alpha_1, \alpha_2) = (3, 6.2)$		
(10, 10)	RB	-0.4990	-0.2168	-0.1643	-0.3851	-0.0629	-0.0338
	RRMSE	0.5459	0.3119	0.2715	0.4606	0.3105	0.2913
	EFR	1.0000	1.7498	2.0103	1.0000	1.4831	1.5808
(20, 30)	RB	-0.2614	-0.0452	-0.0192	-0.2632	-0.0778	-0.0735
	RRMSE	0.2885	0.1021	0.0741	0.0948	0.1569	0.1536
	EFR	1.0000	2.8255	3.8900	1.0000	1.9431	1.9476
(30, 30)	RB	-0.2249	-0.0317	-0.0180	-0.1811	-0.0056	0.0093
	RRMSE	0.2492	0.0830	0.0726	0.2676	0.2074	0.2079
	EFR	1.0000	3.0008	3.4316	1.0000	1.2903	1.2869
(50, 50)	RB	-0.1534	-0.0319	-0.0254	-0.1356	-0.0300	-0.0242
	RRMSE	0.1775	0.0780	0.0721	0.1854	0.1188	0.1167
	EFR	1.0000	2.2765	2.4599	1.0000	1.5609	1.5880
(100, 200)	RB	-00926	-0.0622	-0.0056	-0.0844	-0.0194	-0.0130
	RRMSE	0.1019	0.0350	0.0309	0.1228	0.0814	0.0801
	EFR	1.0000	2.9058	3.2902	1.0000	1.5084	1.5316
	$\lambda_{\text{exact}} = 0.5376$, $(\alpha_1, \alpha_2) = (1, 1.8)$				$\lambda_{\text{exact}} = 0.1108$, $(\alpha_1, \alpha_2) = (1, 3.5)$		
(10, 10)	RB	-0.3198	-0.0512	0.0106	-0.1758	-0.0918	-0.1069
	RRMSE	0.3947	0.3395	0.3180	0.5833	0.6024	0.5358
	EFR	1.0000	1.1625	1.2411	1.0000	0.9682	1.0886
(20, 30)	RB	-0.1933	-0.0061	0.0185	0.0046	0.0535	0.0662
	RRMSE	0.3274	0.2863	0.2597	0.5074	0.5301	0.4983
	EFR	1.0000	1.1434	1.2603	1.0000	0.9573	1.1083
(30, 30)	RB	-0.1676	0.0034	0.0105	0.0176	0.0697	0.1152
	RRMSE	0.3316	0.2405	0.2297	0.5000	0.5271	0.4781
	EFR	1.0000	1.3791	1..4436	1.0000	0.9486	1.0458
(50, 50)	RB	-0.1330	-0.0149	-0.0008	-0.0018	0.0125	-0.0025
	RRMSE	0.2415	0.1803	0.1830	0.4377	0.4521	0.4070
	EFR	1.0000	1.3388	1.3193	1.0000	0.9682	1.0754
(100, 200)	RB	-0.0686	-0.0059	-0.0039	0.0218	0.0348	0.0312
	RRMSE	0.1339	0.1011	0.0947	0.2363	0.2076	0.2102
	EFR	1.0000	1.3242	1.4138	1.0000	1.1381	1.1241

Table 3 The RB, RRMSE and EFR of the two estimators $\hat{\lambda}_k$, and $\hat{\lambda}_{p1}$ when the data are simulated from pair Weibull distributions with different scale and different shape parameters

(n_1, n_2)	Measures	$\hat{\lambda}_k$	$\hat{\lambda}_{p1}$	$\hat{\lambda}_k$	$\hat{\lambda}_{p1}$
		$\lambda_{\text{exact}} = 0.9721$		$\lambda_{\text{exact}} = 0.7889$	
		$(\alpha_1, \alpha_2) = (1,1.2)$, $(\beta_1, \beta_2) = (2, 1)$		$(\alpha_1, \alpha_2) = (1,1.5)$, $(\beta_1, \beta_2) = (3\ 1.9)$	
(10, 10)	RB	-0.3299	-0.1282	-0.3374	-0.0193
	RRMSE	0.3697	0.1908	0.4246	0.1737
	EFR	1.0000	1.9376	1.0000	2.4447
(20, 30)	RB	-0.2306	-0.0626	-0.2235	-0.0313
	RRMSE	0.2688	0.1217	0.2795	0.1421
	EFR	1.0000	2.2091	1.0000	1.9667
(30, 30)	RB	-0.1661	-0.0510	-0.2111	-0.0503
	RRMSE	0.1912	0.0817	0.2604	0.1427

(n_1, n_2)	Measures	$\hat{\lambda}_k$	$\hat{\lambda}_{p1}$	$\hat{\lambda}_k$	$\hat{\lambda}_{p1}$
	EFR	1.0000	2.3397	1.0000	1.8249
(50, 50)	RB	-0.0932	-0.0059	-0.1359	-0.0224
	RRMSE	0.1191	0.0389	0.1749	0.0946
	EFR	1.0000	3.0572	1.0000	1.8478
(100, 200)	RB	-0.0386	-0.0051	-0.0880	-0.0179
	RRMSE	0.0472	0.0229	0.1064	0.0546
	EFR	1.0000	2.0587	1.0000	1.9485
		$\lambda_{exact} = 0.5672$		$\lambda_{exact} = 0.2069$	
		$(\alpha_1, \alpha_2) = (1,1.8)$, $(\beta_1, \beta_2) = (4, 2.1)$		$(\alpha_1, \alpha_2) = (1,3)$, $(\beta_1, \beta_2) = (6, 2)$	
(10, 10)	RB	-0.5198	-0.1871	-0.6151	-0.1948
	RRMSE	0.5767	0.4059	0.7018	0.5759
	EFR	1.0000	1.4209	1.0000	1.2186
(20, 30)	RB	-0.2491	-0.0185	-0.3666	0.0750
	RRMSE	0.3178	0.2030	0.5075	0.3916
	EFR	1.0000	1.5649	1.0000	1.2958
(30, 30)	RB	-0.2430	-0.0318	-0.4242	-0.0656
	RRMSE	0.3203	0.1886	0.5052	0.4017
	EFR	1.0000	1.6984	1.0000	1.2575
(50, 50)	RB	-0.2017	-0.0293	-0.2558	0.0088
	RRMSE	0.2544	0.1723	0.3549	0.2736
	EFR	1.0000	1.4767	1.0000	1.2969
(100, 200)	RB	-0.1135	-0.0037	-0.2097	-0.0435
	RRMSE	0.1574	0.0880	0.2363	0.1284
	EFR	1.0000	1.7878	1.0000	1.8398

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