Point Estimation of Pianka and Kullback-Leibler Overlapping Coefficients: Two Normal Distributions

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Abstract: The well-known two overlapping coefficients; Pianka and Kullback-Leibler are considered to be important measures that focus on measuring the closeness or similarity between two continuous or discrete probability distributions. A general expression for each of them has been developed for some statistical distributions, such as exponential distribution and Weibull distribution. In this research, we focused on the most famous statistical distribution, which is the normal probability distribution.Our main contribution in this paper is to derivation and estimation the two overlapping coefficients Pianka (PI) and Kullback-Leibler (KL) under the assumption of existing a pair of normal distributions. The mathematical formulas (parameters) for each of the PI and KL coefficients were obtained without imposing any restrictions on the parameters of the pair normal distributions. To complete the estimation process of each resulting parameter, an estimator was proposed for each of them by deriving the maximum likelihood estimator based on the assumption of existing two independent random samples, each following the statistical normal distribution.

Keywords: Overlapping Coefficient; Pianka Coefficient; Kullback-Leibler Coefficient; Maximum Likelihood Estimation Method; Normal Distribution

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1 Introduction

The importance of overlapping (OVL) coefficients comes from their potential use and application in many fields, such as environment [1], genetic [2] and reliability analysis [3]. Also, in recent years, it has become important in the goodness of test (see [4, 5]). There are five known and important OVL coefficients, which are Matusita (ρ), Morisita (λ), Weitzman (Δ), Pianka (PI) and Kullback-Leibler (KL) (see [6-11]). These coefficients measure the similarity or agreement between two distributions. Referring to the literature, we find that researchers' interest has focused on estimating these coefficients using two methods: the parametric method (see [12-20]), which assumes a pair of known distributions but with unknown parameters and the non-parametric method (see [6, 21-25]), which assumes that the pair of distributions under study is unknown. There is a greater focus from researchers on studying the first three OVL coefficients, but the study of the other coefficients has not received the same attention, even though they are no less important than the other coefficients. In this study, we focused our attention on examining the Pianka and Kullback-Leibler coefficients, assuming the existence of two normal distributions. A research [26] studied Kullback-Leibler coefficients under pair exponential distributions and research [9] examined the two coefficient under pair Weibull distributions.

Let X and Y be two independent continuous random variables follow $f_X(x;\mu_1,\sigma_1^2)=N(\mu_1,\sigma_1^2)$ and

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$f_Y(x;\mu_2,\sigma_2^2)=N(\mu_2,\sigma_2^2)$ respectively, where $N(\mu,\sigma^2)$ is the pdf of a normal distribution with mean μ and variance σ^2 then the formulas of these two measures are defined by, Pianka coefficient [9, 26].

$$PI=\frac{\int f_X(x;\mu_1,\sigma_1^2)f_Y(x;\mu_2,\sigma_2^2)\,dx}{\sqrt{\int\left[f_X(x;\mu_1,\sigma_1^2)\right]^2dx\ \int\left[f_Y(x;\mu_2,\sigma_2^2)\right]^2dx}}$$

Kullback-Leibler coefficient [9, 26]

$$KL=\frac{1}{1+\int\left(f_X(x;\mu_1,\sigma_1^2)-f_Y(x;\mu_2,\sigma_2^2)\right)log\left(f_X(x;\mu_1,\sigma_1^2)/f_Y(x;\mu_2,\sigma_2^2)\right)dx}$$

2 Pianka Coefficient and Normal Distributions

We will derive the exact value of Pianka measure (PI) between X and Y without assuming any restrictions on the location parameters or the scale parameters. Define,

$$PI=\frac{\int_{-\infty}^{\infty}f_X(x;\mu_1,\sigma_1^2)f_Y(x;\mu_2,\sigma_2^2)\,dx}{\sqrt{\int_{-\infty}^{\infty}\left[f_x(x;\mu_1,\sigma_1^2)\right]^2dx\int_{-\infty}^{\infty}\left[f_y(x;\mu_2,\sigma_2^2)\right]^2dx}}=\frac{Z_{XY}}{\sqrt{Z_{XX}Z_{YY}}}$$

where,

$$Z_{XY}=\int_{-\infty}^{\infty}f_X(x;\mu_1,\sigma_1^2)f_Y(x;\mu_2,\sigma_2^2)\,dx\,,Z_{XX}=\int_{-\infty}^{\infty}[f_X(x;\mu_1,\sigma_1^2)]^2dx,\ Z_{YY}=\int_{-\infty}^{\infty}[f_Y(x;\mu_2,\sigma_2^2)]^2dx$$

Therefore,

$$Z_{XY}=\frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty}\mathrm{e}^{-(x^2-2x\mu_1+\mu_1^2)/2\sigma_1^2}\,\mathrm{e}^{-(x^2-2x\mu_2+\mu_2^2)/2\sigma_2^2}dx$$

$$=\frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty}\mathrm{e}^{-\left(\frac{x^2}{2\sigma_1^2}\right)+\left(\frac{2x\mu_1}{2\sigma_1^2}\right)-\left(\frac{\mu_1^2}{2\sigma_1^2}\right)-\left(\frac{x^2}{2\sigma_2^2}\right)+\left(\frac{2x\mu_2}{2\sigma_2^2}\right)-\left(\frac{\mu_2^2}{2\sigma_2^2}\right)}dx$$

$$=\frac{1}{2\pi\sigma_1\sigma_2}\mathrm{e}^{-\frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}\right)}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}\left\{\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)x^2-2\left(\frac{\mu_1}{\sigma_1^2}+\frac{\mu_2}{\sigma_2^2}\right)x\right\}}dx$$

To simplify the notations, let

$$C=\frac{1}{2\pi\sigma_1\sigma_2}\mathrm{e}^{-\frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}\right)},R_1=\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\text{ and }R_2=\frac{\mu_1}{\sigma_1^2}+\frac{\mu_2}{\sigma_2^2},$$

then,

$$Z_{XY}=C\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}\{R_1x^2-2R_2x\}}dx=C\mathrm{e}^{\frac{R_1}{2}\left(\frac{R_2}{R_1}\right)^2}\frac{\sqrt{2\pi}}{\sqrt{R_1}}\int_{-\infty}^{\infty}\frac{\sqrt{R_1}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{R_1}{2}\left(x-\frac{R_2}{R_1}\right)^2}dx=C\frac{\sqrt{2\pi}}{\sqrt{R_1}}\mathrm{e}^{\frac{R_2{}^2}{2R_1}}$$

After some simplification, we get,

$$Z_{XY}=\frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\mathrm{e}^{-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}}$$

The other two quantities Z_{XX} and Z_{YY} can be counting as follows,

$$Z_{XX}=\int_{-\infty}^{\infty}[f_x(x;\mu_1,\sigma_1^2)]^2dx=\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^2\int_{-\infty}^{\infty}e^{-\frac{2}{2\sigma_1^2}(x-\mu_1)^2}dx$$

$$= \left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^2 \frac{\sigma_1\sqrt{2\pi}}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sigma_1\sqrt{2\pi}} e^{-\frac{2}{2\sigma_1^2}(x-\mu_1)^2} dx = \frac{1}{2\sigma_1\sqrt{\pi}}$$

If the last result is $Z_{XX} = \frac{1}{2\sigma_1\sqrt{\pi}}$ is considered to be a function of μ_1 and σ_1^2 (say, $\psi(\mu_1, \sigma_1^2)$) then

$$Z_{YY} = \psi(\mu_2, \sigma_2^2) = \frac{1}{2\sigma_2\sqrt{\pi}}.$$

Thus,

$$\sqrt{Z_{XX}Z_{YY}} = \frac{1}{2\sqrt{\pi\sigma_1\sigma_2}}.$$

Finally, the formula of OVL Pianka coefficient (PI) is,

$$PI = \frac{Z_{XY}}{\sqrt{Z_{XX}Z_{YY}}} = \frac{\sqrt{2\sigma_1\sigma_2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}} \mathrm{e}^{-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}}$$

On one hand, if we assume that $\sigma_1 = \sigma_2$ ($= \sigma$, say) then,

$$PI = \mathrm{e}^{-\left(\frac{\mu_1-\mu_2}{2\sigma}\right)^2}.$$

On the other hand, if we assume that $\mu_1 = \mu_2$ ($= \mu$, say) then,

$$PI = \frac{\sqrt{2\sigma_1\sigma_2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}}.$$

In addition, if $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$ then,

$$PI = 1.$$

3 Kullback-Leibler Coefficient and Normal Distributions

In this section is interest to find the exact formula of Kullback-Leibler Measure (KL) under the assumption $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$. In this case, the formula of KL is given by (See Section 1.2),

$$KL = \frac{1}{1+\int_{-\infty}^{\infty}\left(f_X(x;\mu_1,\sigma_1^2)-f_Y(x;\mu_2,\sigma_2^2)\right) log\left(\frac{f_X(x;\mu_1,\sigma_1^2)}{f_Y(x;\mu_2,\sigma_2^2)}\right) dx}$$

The quantity inside the integral in the denominator of the above formula can be simplify as given below,
Therefore, we can write the integral as follows,

$$\int_{-\infty}^{\infty}\left(f_X(x;\mu_1,\sigma_1^2) - f_Y(x;\mu_2,\sigma_2^2)\right) log\left(\frac{f_X(x;\mu_1,\sigma_1^2)}{f_Y(x;\mu_2,\sigma_2^2)}\right) dx = a - b - c + d$$

where,

$$a = \int_{-\infty}^{\infty} f_X(x;\mu_1,\sigma_1^2)\, log f_X(x;\mu_1,\sigma_1^2) dx$$

$$b = \int_{-\infty}^{\infty} f_X(x;\mu_1,\sigma_1^2)\, log f_Y(x;\mu_2,\sigma_2^2) dx$$

$$c = \int_{-\infty}^{\infty} f_Y(x;\mu_2,\sigma_2^2)\, log f_X(x;\mu_1,\sigma_1^2) dx$$

and

$$d = \int_{-\infty}^{\infty} f_Y(x;\mu_2,\sigma_2^2) log f_Y(x;\mu_2,\sigma_2^2)dx$$

if $a = w(\mu_1,\sigma_1^2)$ then $d = w(\mu_2,\sigma_2^2)$. Also, if $b = l(\mu_1,\sigma_1^2,\mu_2,\sigma_2^2)$ then $c = l(\mu_2,\sigma_2^2,\mu_1,\sigma_1^2)$. Therefore, it is enough to find a and b or c and d. Now,

$$a = \int_{-\infty}^{\infty} f_x(x;\mu_1,\sigma_1^2)\ log f_x(x;\mu_1,\sigma_1^2)dx$$

$$= -\log\sqrt{2\pi\sigma_1^2} - \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\left(\frac{x-\mu_1}{\sigma_1}\right)^2} dx$$

$$= -\log\sqrt{2\pi\sigma_1^2} - \frac{1}{2}E\left(\frac{X-\mu_1}{\sigma_1}\right)^2$$

$$= -\log\sqrt{2\pi\sigma_1^2} - \frac{1}{2\sigma_1^2}Var(X)$$

$$= -\log\sqrt{2\pi\sigma_1^2} - \frac{1}{2}$$

Therefore, $d = -\log\sqrt{2\pi\sigma_2^2} - \frac{1}{2}$. Now,

$$b = -log\sqrt{2\pi\sigma_2^2} - \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} . \frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2 dx$$

$$= -log\sqrt{2\pi\sigma_2^2} - \frac{1}{2\sigma_2^2}E(X-\mu_2)^2$$

$$= -log\sqrt{2\pi\sigma_2^2} - \frac{1}{2\sigma_2^2}(EX^2 - 2\mu_2 EX + \mu_2^2)$$

$$= -log\sqrt{2\pi\sigma_2^2} - \frac{1}{2\sigma_2^2}(\sigma_1^2 + (\mu_1-\mu_2)^2).$$

Therefore, $c = -log\sqrt{2\pi\sigma_1^2} - \frac{1}{2\sigma_1^2}(\sigma_2^2 + (\mu_2-\mu_1)^2)$ and

$$a - b - c + d = \frac{1}{2\sigma_2^2}(\sigma_1^2 + (\mu_1-\mu_2)^2) + \frac{1}{2\sigma_1^2}(\sigma_2^2 + (\mu_2-\mu_1)^2) - 1.$$

Finally, after some simple calculations we obtain the value of Kullback-Leibler (KL) coefficient, which is given by,

$$KL = \frac{1}{\frac{1}{2\sigma_2^2}(\sigma_1^2+(\mu_1-\mu_2)^2)+\frac{1}{2\sigma_1^2}(\sigma_2^2+(\mu_2-\mu_1)^2)} = \frac{2}{\frac{\sigma_1^2}{\sigma_2^2}+\frac{\sigma_2^2}{\sigma_1^2}+(\mu_1-\mu_2)^2\left(\frac{1}{\sigma_2^2}+\frac{1}{\sigma_1^2}\right)}$$

Again, if $\sigma_1 = \sigma_2 = \sigma$, then,

$$KL = \frac{2}{2+\left(\frac{2}{\sigma^2}\right)(\mu_1-\mu_2)^2},$$

but if $\mu_1 = \mu_2$, then,

$$KL = \frac{2}{\frac{\sigma_1^2}{\sigma_2^2}+\frac{\sigma_2^2}{\sigma_1^2}}$$

Also, if $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$ then,

$$KL = 1$$

4 ML Estimators of Pianka and Kullback-Leibler Coefficients

Let $f_X(x;\hat{\mu}_1,\hat{\sigma}_1^2)$ and $f_Y(x;\hat{\mu}_2,\hat{\sigma}_1^2)$ be the maximum likelihood (ML) estimators of $f_X(x;\mu_1,\sigma_1^2)$ and $f_Y(x;\mu_2,\sigma_2^2)$ where $X_1,X_2,\dots,X_{n_1}$ be a random sample of size n_1 from $N(\mu_1,\sigma_1^2)$ and $Y_1,Y_2,\dots,Y_{n_2}$ be another random sample of size n_2 from $N(\mu_2,\sigma_2^2)$, where the two samples are independent. To find the ML estimators of $\mu_1,\mu_2,\ \sigma_1^2$and σ_2^2.

The joint pdf of $X_1,X_2,\dots,X_{n_1}$,

$$f_X\left(x_1,x_2,\dots,x_{n_1};\ \mu_1,\sigma_1^2\right)=\left(\frac{1}{2\pi\sigma_1^2}\right)^{n_1/2}e^{-\frac{1}{2}\sum_{i=1}^{n_1}\left(\frac{x_i-\mu_1}{\sigma_1}\right)^2}$$

and the joint pdf of $Y_1,Y_2,\dots,Y_{n_2}$ is,

$$f_Y\left(y_1,y_2,\dots,y_{n_2};\ \mu_2,\sigma_2^2\right)=\left(\frac{1}{2\pi\sigma_2^2}\right)^{n_2/2}e^{-\frac{1}{2}\sum_{i=1}^{n_2}\left(\frac{y_i-\mu_2}{\sigma_2}\right)^2}.$$

Therefore, the likelihood function of $\mu_1,\mu_2,\sigma_1^2,\sigma_2^2$ is,

$$L(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)=\left(\frac{1}{2\pi\sigma_1^2}\right)^{n_1/2}\left(\frac{1}{2\pi\sigma_2^2}\right)^{n_2/2}e^{-\frac{1}{2}\sum_{i=1}^{n_1}\left(\frac{x_i-\mu_1}{\sigma_1}\right)^2-\frac{1}{2}\sum_{i=1}^{n_2}\left(\frac{y_i-\mu_2}{\sigma_2}\right)^2}$$

and the natural logarithm of $L(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)$ is,

$$lnL(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)=-\frac{n_1}{2}\ln\sigma_1^2-\frac{n_2}{2}\ln\sigma_2^2-\frac{1}{2}\sum_{i=1}^{n_1}\left(\frac{x_i-\mu_1}{\sigma_1}\right)^2-\frac{1}{2}\sum_{i=1}^{n_2}\left(\frac{y_i-\mu_2}{\sigma_2}\right)^2$$

By finding the derivatives of $lnL(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2)$ with respect to each $\mu_1,\mu_2,\ \sigma_1^2$and σ_2^2 and equating to zero, yields the ML estimators of $\mu_1,\ \mu_2,\ \sigma_1^2$ and σ_2^2, which are respectively given by, $\hat{\mu}_1=\bar{X},\hat{\mu}_2=\bar{Y},\hat{\sigma}_1^2=S_1^2$ and $\hat{\sigma}_2^2=S_2^2$, where, $\bar{X}=\sum_{j=1}^{n_1}X_j/n_1$, $\bar{Y}=\sum_{j=1}^{n_2}Y_j/n_2$, $S_1^2=\sum_{j=1}^{n_1}(X_j-\bar{X})^2/n_1$ and $S_2^2=\sum_{j=1}^{n_2}(Y_j-\bar{Y})^2/n_2$.

Therefore, the ML estimators of the two OVL measures PI and KL are respectively,

$$\widehat{PI}=\frac{\sqrt{2S_1S_2}}{\sqrt{(S_1^2+S_2^2)}}\,\mathrm{e}^{-\frac{(\bar{X}-\bar{Y})^2}{2(S_1^2+S_2^2)}},$$

and

$$\widehat{KL}=\frac{2}{\frac{S_1^2}{S_2^2}+\frac{S_2^2}{S_1^2}+(\bar{X}-\bar{Y})^2\left(\frac{1}{S_2^2}+\frac{1}{S_1^2}\right)}.$$

5 Discussion

After finding the corresponding integral value for each of the PI and KL overlapping coefficients and presenting it in an closed form, we proposed the ML estimator for each overlapping coefficient. As it is well known, the ML estimator is good estimator and is at least a consistent estimator as the sample size increases. That is, it is unbiased and its variance approaches zero as the sample size increases. Also, asymptotic distribution of the ML estimator can be found as the sample sizes increase. Based on these asymptotic distributions, confidence intervals can be established for each of the PI and KL overlapping coefficients. This topic is currently under study as an idea for an upcoming research project.

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