Further Refinements of Shafer–Fink Type Inequalities for Lemniscatic Functions

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Abstract: The lemniscatic functions and their inverses play a vital role in various fields of applied science. In classical mechanics, they appear in the exact solutions of the nonlinear pendulum equation and describe the trajectories of particles in central force fields. In electromagnetism, they are involved in the modeling of electric and magnetic field distributions with elliptic symmetry. Moreover, in engineering mathematics and optical system design, lemniscatic functions arise in problems involving wave propagation, signal transmission, and conformal mappings. Their mathematical properties make them valuable tools in both theoretical analysis and computational modeling. Lemniscatic functions, such as Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine, arise from the incompletely symmetric elliptic integral of the first kind. In 2007, Neuman introduced additional lemniscatic functions including the arc lemniscate tangent and its hyperbolic counterpart. These functions have attracted considerable attention due to their deep connections with elliptic integrals and their applications in inequality theory. In this paper, we build upon recent work by Wei, He, and Wang (2020), who established Shafer–Fink type inequalities for these lemniscatic functions. Employing methods from real analysis, we present new and sharper refinements of these inequalities. Specifically, we establish double inequalities for Gauss' arc lemniscate sine, the hyperbolic arc lemniscate sine, arc lemniscate tangent and its hyperbolic counterpart bounded by rational expressions involving radical terms. These results not only improve the existing Shafer–Fink type inequalities but also provide further insight into the monotonicity and convexity properties of the involved lemniscatic functions. Our approach offers a unified framework for deriving tight bounds for all four Neuman lemniscatic functions, contributing to the broader understanding of special function inequalities. These results represent significant refinements over previous known estimates and open avenues for further generalizations.

Keywords: Arc Lemniscatic Function; Hyperbolic Arc Lemniscate Function; Shafer–Fink Type Inequalities

DOI: 10.57237/j.wjms.2025.02.001

1 Introduction

For $x,y,z\ge 0$ with $xy+xz+yz\ne 0$, the incompletely symmetric elliptic integral of the first kind $R_F(x,y,z)$ [1] is defined by $R_F(x,y,z)=\frac{1}{2}\int_0^\infty\left[(t+x)(t+y)(t+z)\right]^{-1/2}dt$.

The Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine functions are defined as follows (see [2,

Funding: The Key Project of the Scientific Research of Zhejiang Open University in 2019 (Grant no: XKT- 19Z02);
The Natural Science Foundation of the Department of Education of Zhejiang Province in 2020 (Grant noY202043179).

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Received: 21 April 2025; Accepted: 1 July 2025; Published Online: 19 July 2025 http://www.wjoms.com

3]):

$$arcsl(x)=\int_0^x \frac{dt}{\sqrt{1-t^4}},\quad |x|\le 1 \tag{1}$$

and

$$arcslh(x)=\int_0^x \frac{dt}{\sqrt{1+t^4}},\quad x\in \mathbb{R} \tag{2}$$

respectively.

It was shown by Carlson in [1, Ex. 8.1-5, 8.3-7] that

$$arcsl(x)=xR_F\left(1-x^2,1+x^2,1\right),$$
$$arcslh(x)=xR_F\left(1-ix^2,1+ix^2,1\right).$$

Moreover, the arc lemniscate sine function has a simple geometric interpretation: it represents the arc length measured from the origin to a point with polar coordinates on the Bernoulli lemniscate $r^2=\cos 2\theta$ is $arcsl(r)$.

Another pair of arc lemniscate functions, the Gauss' arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions are defined in terms of the arc lemniscate sine and the hyperbolic arc lemniscate sine functions, respectively (see [4]):

$$arctl(x)=arcsl\left(\frac{x}{\sqrt[4]{1+x^4}}\right),\quad x\in \mathbb{R} \tag{3}$$

and

$$arctlh(x)=arcslh\left(\frac{x}{\sqrt[4]{1-x^4}}\right),\quad |x|<1. \tag{4}$$

Further, the first lemniscate constant (c. f. [2], Theorem 1.7) and [7]) is given by

$$\omega=arcsl(1)=\frac{1}{\sqrt{2}}\kappa\left(1/\sqrt{2}\right)=\frac{\Gamma^2(1/4)}{4\sqrt{2\pi}}=1.31103\ldots$$

And some related constants that will be used later

$$\sigma=arcslh(1)=\omega/\sqrt{2}=0.92703\ldots,$$
$$\kappa=arcslh(+\infty)=\sqrt{2}\omega=1.85407\ldots,$$

$$\tau=arctl(1)=arcsl\left(1/\sqrt[4]{2}\right)=0.89558\ldots,$$

where

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

is the classical Euler gamma function [8-10] and

$$\kappa(r)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-r^2\sin^2(\theta)}}=R_F\left(0,1-r^2,1\right),\quad (0<r<1)$$

is the complete elliptic integral of the first kind [11-15].

In recent few years, approximations of the arc lemniscatic functions have attracted the attention of many researchers. In particular, many remarkable properties and inequalities related to these functions can be found in the literature [5, 16-20].

Recently, Neuman [6] presented the simple lower and upper bounds for the lemniscate functions as follows:

$$\left(\frac{5}{3+2\sqrt{1-x^4}}\right)^{1/2}<\frac{arcsl(x)}{x}<\left(1-x^4\right)^{-1/10},$$

$$\left(\frac{5}{3+2\sqrt{1+x^4}}\right)^{1/2}<\frac{arcslh(x)}{x}<\left(1+x^4\right)^{-1/10},$$

$$\left(\frac{5}{3\sqrt{1-x^4}+2}\right)^{1/2}<\frac{arctlh(x)}{x}<\left(1-x^4\right)^{-3/20},$$

$$\left(\frac{5}{3\sqrt{1+x^4}+2}\right)^{1/2}<\frac{arctl(x)}{x}<\left(1+x^4\right)^{-3/20}$$

for all $0<|x|<1$.

Wei et al. [21] proved the Shafer–Fink type inequalities.

$$\frac{5}{4+\sqrt{1-x^4}}<\frac{arcsl(x)}{x}<\frac{5}{3+2\sqrt[4]{1-x^4}}, \tag{5}$$

$$\frac{5}{4+\sqrt{1+x^4}}<\frac{arcslh(x)}{x}<\frac{5}{3+2\sqrt[4]{1+x^4}}, \tag{6}$$

$$\frac{10}{7+3\sqrt{1-x^4}}<\frac{arctlh(x)}{x}<\frac{5}{2+3\sqrt[4]{1-x^4}}, \tag{7}$$

$$\frac{10}{7+3\sqrt{1+x^4}}<\frac{arctl(x)}{x}<\frac{5}{2+3\sqrt[4]{1+x^4}} \tag{8}$$

for all $0<|x|<1$.

Motivated by inequalities (5)–(8), we continue our study of the so-called Shafer–Fink type inequalities in the context of arc lemniscatic functions. The main purpose of the article is to present the best possible parameters $\lambda_1,\lambda_2,\lambda_3,\ \lambda_4,\mu_1,\mu_2,\mu_3$ and μ_4 such that the double inequalities.

$$\frac{1}{\lambda_1+2(4/5-\lambda_1)\sqrt[4]{1-x^4}+(\lambda_1-3/5)\sqrt{1-x^4}}<\frac{arcsl(x)}{x}$$

$$<\frac{1}{\mu_1+2(4/5-\mu_1)\sqrt[4]{1-x^4}+(\mu_1-3/5)\sqrt{1-x^4}},$$

$$\frac{1}{\lambda_2+2(4/5-\lambda_2)\sqrt[4]{1+x^4}+(\lambda_2-3/5)\sqrt{1+x^4}}<\frac{arcslh(x)}{x}$$

$$<\frac{1}{\mu_2+2(4/5-\mu_2)\sqrt[4]{1+x^4}+(\mu_2-3/5)\sqrt{1+x^4}},$$

$$\frac{1}{\lambda_3+(7/5-2\lambda_3)\sqrt[4]{1-x^4}+(\lambda_3-2/5)\sqrt{1-x^4}}<\frac{arctlh(x)}{x}$$

$$<\frac{1}{\mu_3+(7/5-2\mu_3)\sqrt[4]{1-x^4}+(\mu_3-2/5)\sqrt{1-x^4}},$$

$$\frac{1}{\lambda_4+(7/5-2\lambda_4)\sqrt[4]{1+x^4}+(\lambda_4-2/5)\sqrt{1+x^4}}<\frac{arctl(x)}{x}$$

$$<\frac{1}{\mu_4+(7/5-2\mu_4)\sqrt[4]{1+x^4}+(\mu_4-2/5)\sqrt{1+x^4}}$$

for all $0<|x|<1$.

2 Lemmas

To prove our main results, we first introduce some basic knowledge of arc lemniscate functions along with two essential lemmas, which are presented in this section. Using their definitions and the chain rule, we recall the derivative formulas of the arc lemniscate functions as follows:

$$\frac{d\left[arcsl(x)\right]}{dx}=\left(1-x^4\right)^{-1/2},\quad \frac{d\left[arctlh(x)\right]}{dx}=\left(1-x^4\right)^{-3/4},\quad |x|<1.$$

$$\frac{d\left[arcslh(x)\right]}{dx}=\left(1+x^4\right)^{-1/2},\quad \frac{d\left[arctl(x)\right]}{dx}=\left(1+x^4\right)^{-3/4},\quad x\in\mathbb{R}.$$

Lemma 2.1 Let $p_0=\left[5-\sqrt[4]{2}\left(8-3\sqrt[4]{2}\right)\sigma\right]/\left[5\left(1-\sqrt[4]{2}\right)^2\sigma\right]=0.6844\ldots$, $p\in(0,1)$ and

$$f(t)=\left(25p^2-5p-6\right)t^3-10\left(5p^2-13p+6\right)t^2+25\left(p^2+2p-2\right)t+10\left(5p-4\right)\tag{9}$$

Then the following statements are true:

(1) If $p=52/75$, then $f(t)<0$ for all $t\in(0,1)$ and $f(t)>0$ for all $t\in\left(1,\sqrt[4]{2}\right)$;

(2) If $p=1/\omega$, then there exists $\eta\in(0,1)$ such that $f(t)<0$ for $t\in(0,\eta)$ and $f(t)>0$ for $t\in(\eta,1)$;

(3) If $p = p_0$, then there exists $\varsigma \in \left(1, \sqrt[4]{2}\right)$ such that $f(t) < 0$ for $t \in (1, \varsigma)$ and $f(t) > 0$ for $t \in \left(\varsigma, \sqrt[4]{2}\right)$.

Proof (1) If $p = 52/75$, then (9) becomes

$$f(t) = \frac{2}{225}(t-1)\left(287t^2 + 973t + 600\right) \tag{10}$$

Therefore, part (1) follows easily from (10).

(2) If $p = 1/\omega$, then numerical computations lead to

$$25p^2 - 5p - 6 = 4.7312\cdots > 0,\ 5p^2 - 13p + 6 = -1.0068\cdots < 0, \tag{11}$$

$$p^2 + 2p - 2 = 0.1073\cdots > 0,\ 5p - 4 = -0.1862\cdots < 0, \tag{12}$$

$$225p - 156 = 15.6209\cdots > 0,\quad 295p - 188 = 37.0141\cdots > 0. \tag{13}$$

It follows from (9) and (11)-(13) that

$$f(0) = 10(5p-4) < 0,\ f(1) = 295p - 188 > 0, \tag{14}$$

$$f'(t) = 3\left(25p^2 - 5p - 6\right)t^2 - 20\left(5p^2 - 13p + 6\right)t + 25\left(p^2 + 2p - 2\right) > (295p - 188)t^2 > 0 \tag{15}$$

for $t \in (0,1)$.

Therefore, part (2) follows easily from (14) and (15).

(3) If $p = p_0$, then numerical computations lead to

$$25p^2 - 5p - 6 = 2.2887\cdots > 0,\ 5p^2 - 13p + 6 = -0.5553\cdots < 0, \tag{16}$$

$$p^2 + 2p - 2 = -0.1627\cdots < 0,\ 295p - 188 = 13.9053\cdots > 0. \tag{17}$$

It follows from (9) and (16)-(17) that

$$f(1) = 295p - 188 = 13.9053\cdots > 0, \tag{18}$$

$$\begin{aligned} f\left(\sqrt[4]{2}\right) &= 2^{3/4}\left(25p^2 - 5p - 6\right) - 10\sqrt{2}\left(5p^2 - 13p + 6\right) \\ &\quad + 25\sqrt[4]{2}\left(p^2 + 2p - 2\right) + 10(5p - 4) = 1.0867\cdots > 0, \end{aligned} \tag{19}$$

$$\begin{aligned} f(t) &= 3\left(25p^2 - 5p - 6\right)t^2 - 20\left(5p^2 - 13p + 6\right)t + 25\left(p^2 + 2p - 2\right) \\ &> (295p - 188) > 0 \end{aligned} \tag{20}$$

for $t \in \left(1, \sqrt[4]{2}\right)$.

Therefore, part (3) follows easily from (18)-(20).

Lemma 2.2 Let $p_1 = \left[5 - \sqrt[4]{2}\left(7 - 2\sqrt[4]{2}\right)\tau\right] / \left[5\left(1 - \sqrt[4]{2}\right)^2 \tau\right] = 0.4857\cdots$, $p \in (0,1)$ and

$$g(t) = 5(5p-2)t^3 + \left(25p^2 + 30p - 16\right)t^2 - 10\left(5p^2 - 12p + 5\right)t + 5\left(5p^2 + 10p - 7\right) \tag{21}$$

Then the following statements are true:

(1) If $p = 37/75$, then $g(t) < 0$ for all $t \in (0,1)$ and $g(t) > 0$ for all $t \in \left(1, \sqrt[4]{2}\right)$;

(2) If $p = 1/\kappa$, then there exists $\xi \in (0,1)$ such that $g(t) < 0$ for $t \in (0, \xi)$ and $g(t) > 0$ for $t \in (\xi, 1)$;

(3) If $p = p_1$, then there exists $\zeta \in \left(1, \sqrt[4]{2}\right)$ such that $g(t) < 0$ for $t \in (1, \zeta)$ and $g(t) > 0$ for $t \in \left(\zeta, \sqrt[4]{2}\right)$.

Proof (1) If $p = 37/75$, then (21) becomes

$$g(t) = \frac{1}{225}(t-1)\left(525t^2 + 1624t + 956\right) \tag{22}$$

Therefore, part (1) follows easily from (22).

(2) If $p = 1/\kappa$, then numerical computations lead to

$$5p - 2 = 0.6967\mathrm{L} > 0,\ 25p^2 + 30p - 16 = 7.4531\mathrm{L} > 0, \tag{23}$$

$$5p^2 - 12p + 5 = -0.0177\mathrm{L} < 0,\ 5p^2 + 10p - 7 = -0.1519\mathrm{L} < 0, \tag{24}$$

$$225p - 111 = 10.3543\mathrm{L} > 0,\ 255p - 112 = 25.5349\mathrm{L} > 0. \tag{25}$$

It follows from (21) and (23)-(25) that

$$g(0) = 5\left(5p^2 + 10p - 7\right) < 0,\ g(1) = 225p - 111 > 0, \tag{26}$$

$$\begin{aligned} g'(t) &= 15(5p-2)t^2 + 2\left(25p^2 + 30p - 16\right)t - 10\left(5p^2 - 12p + 5\right) \\ &> (255p - 112)t^2 > 0 \end{aligned} \tag{27}$$

for $t \in (0,1)$.

Therefore, part (2) follows easily from (26) and (27).

(3) If $p = p_1$, then numerical computations lead to

$$5p - 2 = 0.4288\mathrm{L} > 0,\ 25p^2 + 30p - 16 = 4.4724\mathrm{L} > 0, \tag{28}$$

$$5p^2 - 12p + 5 = 0.3506\mathrm{L} > 0,\ 255p - 112 = 11.8717\mathrm{L} > 0. \tag{29}$$

It follows from (21) and (28)-(29) that

$$g(1) = 225p - 111 = -1.7014\mathrm{L} < 0, \tag{30}$$

$$\begin{aligned} g\left(\sqrt[4]{2}\right) &= 25\left(1 - \sqrt[4]{2}\right)^2 p^2 + 5\left(5\sqrt[4]{8} + 6\sqrt{2} + 24\sqrt[4]{2} + 10\right)p \\ &\quad - \left(10\sqrt[4]{8} + 16\sqrt{2} + 50\sqrt[4]{2} + 35\right) = 0.9497\mathrm{L} > 0, \end{aligned} \tag{31}$$

$$\begin{aligned} g'(t) &= 15(5p-2)t^2 + 2\left(25p^2 + 30p - 16\right) - 10\left(5p^2 - 12p + 5\right) \\ &> 255p - 112 > 0 \end{aligned} \tag{32}$$

for $t \in \left(1, \sqrt[4]{2}\right)$.

Therefore, part (3) follows easily from (30), (31) and (2.24).

3 Main Results

In this section, we are in a position to state our main results.

Theorem 3.1 The double inequality

$$\frac{1}{\lambda_1+2(4/5-\lambda_1)\sqrt[4]{1-x^4}+(\lambda_1-3/5)\sqrt{1-x^4}}<\frac{arcsl(x)}{x}<\frac{1}{\mu_1+2(4/5-\mu_1)\sqrt[4]{1-x^4}+(\mu_1-3/5)\sqrt{1-x^4}}$$

holds for all $0<|x|<1$ if and only if $\lambda_1\geq 1/\omega=0.7627\ldots$ and $\mu_1\leq 52/75=0.6933\ldots$.

Proof. Without loss of generality, we assume $x\in(0,1)$. Let $p\in(0,1)$ and

$$F_p(x)=arcsl(x)-\frac{x}{p+2(4/5-p)\sqrt[4]{1-x^4}+(p-3/5)\sqrt{1-x^4}}. \tag{33}$$

Then elementary computations lead to

$$\lim_{x\to 0^+}F_p(x)=0, \tag{34}$$

$$\lim_{x\to 1^-}F_p(x)=\omega-1/p, \tag{35}$$

$$F_p'(x)=\frac{\left(1-\sqrt[4]{1-x^4}\right)^2 f\left(\sqrt[4]{1-x^4}\right)}{\left(1-x^4\right)^{3/4}\left[5p+2(4-5p)\sqrt[4]{1-x^4}+(5p-3)\sqrt{1-x^4}\right]^2} \tag{36}$$

where $f(\bullet)$ is defined as in Lemma 2.1.

We divide the proof into three cases.

Case 1.1 $p=52/75$. Then Lemma 2.1 (1), (34) and (36) imply

$$F_p(x)<0 \tag{37}$$

for all $x\in(0,1)$.

Therefore,

$$arcsl(x)<\frac{75x}{52+16\sqrt[4]{1-x^4}+7\sqrt{1-x^4}}$$

for $x\in(0,1)$ follows from (33) and (37).

Case 1.2 $p=1/\omega$. Then Lemma 2.1 (2) and (36) lead to the conclusion that there exists $\eta^*\in(0,1)$ such that $F_p(x)$ is strictly increasing on $(0,\eta^*]$ and strictly decreasing on $[\eta^*,1)$. Note that (35) becomes

$$\lim_{x\to 1^-}F_p(x)=0. \tag{38}$$

It follows from (34) and (38) together with the piecewise monotonicity of $F_p(x)$ that

$$F_p(x)>0 \tag{39}$$

for all $x\in(0,1)$.

Therefore,

$$arcsl(x) > \frac{5\omega x}{5+2(4\omega-5)\sqrt[4]{1-x^4}+(5-3\omega)\sqrt{1-x^4}}$$

for $x\in(0,1)$ follows from (33) and (39).

Case 1.3 $52/75 < p < 1/\omega$. Then it follows from (3.1), by Taylor expansion,

$$\lim_{x\to 0^+}\frac{F_p(x)}{x^9}=\frac{1}{16}\left(p-\frac{52}{75}\right)>0, \tag{40}$$

$$\lim_{x\to 1^-}F_p(x)<0. \tag{41}$$

Equations (33) and inequalities (40) and (41) imply that there exists small enough $0<\delta_1<1$ and large enough $0<\delta_1'<1$ such that

$$arcsl(x) > \frac{5x}{5p+2(4-5p)\sqrt[4]{1-x^4}+(5p-3)\sqrt{1-x^4}}$$

for all $x\in(0,\delta_1)$ and

$$arcsl(x) < \frac{5x}{5p+2(4-5p)\sqrt[4]{1-x^4}+(5p-3)\sqrt{1-x^4}}$$

for all $x\in(1-\delta_1',1)$.

Therefore, Theorem 3.1 follows from Cases 1.1- 1.3 and the monotonicity of $F_p(x)$.

Theorem 3.2 The double inequality

$$\frac{1}{\lambda_2+2(4/5-\lambda_2)\sqrt[4]{1+x^4}+(\lambda_2-3/5)\sqrt{1+x^4}}<\frac{arcslh(x)}{x}<\frac{1}{\mu_2+2(4/5-\mu_2)\sqrt[4]{1+x^4}+(\mu_2-3/5)\sqrt{1+x^4}}$$

hold for all $0<|x|<1$ if and only if $\lambda_2\ge 52/75=0.6933\ldots$ and $\mu_2\le p_0=0.6844\ldots$, where p_0 is defined in Lemma 2.1.

Proof. Without loss of generality, let $p\in(0,1)$, $x\in(0,1)$ and

$$G_p(x)=arcslh(x)-\frac{x}{p+2(4/5-p)\sqrt[4]{1+x^4}+(p-3/5)\sqrt{1+x^4}}. \tag{42}$$

Then simple computations lead to

$$\lim_{x\to 0^+}G_p(x)=0, \tag{43}$$

$$\lim_{x\to 1^-}G_p(x)=\sigma-\frac{5}{5\left(1-\sqrt[4]{2}\right)^2p+4\sqrt{2}\left(8-3\sqrt[4]{2}\right)}, \tag{44}$$

$$G_p'(x)=\frac{\left(\sqrt[4]{1+x^4}-1\right)^2 f\left(\sqrt[4]{1+x^4}\right)}{\left(1+x^4\right)^{3/4}\left(5p+2(4-5p)\sqrt[4]{1+x^4}+(5p-3)\right)^2} \tag{45}$$

where $f(\bullet)$ is defined as in Lemma 2.1.

We divide the proof into three cases.

Case 2.1 $p = 52/75$. Then from Lemma 2.1 (1), (43) and (45) we obtain

$$G_p(x) > 0 \tag{46}$$

for all $x \in (0,1)$.

Therefore,

$$arcslh(x) > \frac{5x}{52 + 16\sqrt[4]{1+x^4} + 7\sqrt{1+x^4}}$$

for $x \in (0,1)$ follows from (42) and (46).

Case 2.2 $p = p_0$. Then Lemma 2.1 (3) and (45) lead to the conclusion that there exists $\varsigma^* \in (0,1)$ such that $G_p(x)$ is strictly decreasing on $(0, \varsigma^*]$ and strictly increasing on $[\varsigma^*, 1)$. Note that (44) becomes

$$\lim_{x \to 1^-} G_p(x) = 0. \tag{47}$$

It follows from (43) and (47) together with the piecewise monotonicity of $G_p(x)$ that

$$G_p(x) < 0 \tag{48}$$

for all $x \in (0,1)$.

Therefore,

$$arcslh(x) < \frac{5x}{5p_0 + 2(4 - 5p_0)\sqrt[4]{1+x^4} + (5p_0 - 3)\sqrt{1+x^4}}$$

for $x \in (0,1)$ follows from (42) and (48).

Case 2.3 $p_0 < p < 52/75$. Then (42) and (44) lead to

$$\lim_{x \to 0^+} \frac{G_p(x)}{x^9} = \frac{1}{16}\left(p - \frac{52}{75}\right) < 0, \tag{49}$$

$$\lim_{x \to 1^-} G_p(x) > 0. \tag{50}$$

Equations (42) and inequalities (49) and (50) imply that there exists small enough $0 < \delta_2 < 1$ and large enough $0 < \delta_2' < 1$ such that

$$arcslh(x) < \frac{5x}{5p + 2(4 - 5p)\sqrt[4]{1+x^4} + (5p - 3)\sqrt{1+x^4}}$$

for all $x \in (0, \delta_2)$ and

$$arcslh(x) > \frac{5x}{5p + 2(4 - 5p)\sqrt[4]{1+x^4} + (5p - 3)\sqrt{1+x^4}}$$

for $x \in (1-\delta_2', 1)$.

Therefore, Theorem 3.2 follows from Cases 2.1- 2.3 and the monotonicity of $G_p(x)$.

Theorem 3.3 The double inequality

$$\frac{1}{\lambda_3 + (7/5 - 2\lambda_3)\sqrt[4]{1-x^4} + (\lambda_3 - 2/5)\sqrt{1-x^4}} < \frac{arctlh(x)}{x}$$

$$< \frac{1}{\mu_3 + (7/5 - 2\mu_3)\sqrt[4]{1-x^4} + (\mu_3 - 2/5)\sqrt{1-x^4}}$$

holds for all $0 < |x| < 1$ if and only if $\lambda_3 \ge 1/\kappa = 0.5393\ldots$ and $\mu_3 \le 37/75 = 0.4933\ldots$.

Proof. Since $\sqrt[4]{1-x^4}$, $\sqrt{1-x^4}$ and $arctlh(x)/x$ are even functions, without loss of generality, we assume $x \in (0,1)$. Let $p \in (0,1)$ and

$$H_p(x) = arctlh(x) - \frac{x}{p + (7/5 - 2p)\sqrt[4]{1-x^4} + (p - 2/5)\sqrt{1-x^4}} \tag{51}$$

Then elaborated computations lead to

$$\lim_{x \to 0^+} H_p(x) = 0, \tag{52}$$

$$\lim_{x \to 1^-} H_p(x) = \kappa - \frac{1}{p}, \tag{53}$$

$$H_p'(x) = \frac{\left(1-\sqrt[4]{1-x^4}\right)^2 g\left(\sqrt[4]{1-x^4}\right)}{\left(1-x^4\right)^{3/4}\left[5p + (7-10p)\sqrt[4]{1-x^4} + (5p-2)\sqrt{1-x^4}\right]^2} \tag{54}$$

where $g(\bullet)$ is defined as in Lemma 2.2.

We divide the proof into three cases.

Case 3.1 $p = 37/75$. Then it can be obtained from Lemma 2.2 (1), (52) and (55) that

$$H_p(x) < 0 \tag{55}$$

for all $x \in (0,1)$.

Therefore,

$$arctlh(x) < \frac{75x}{37 + 31\sqrt[4]{1-x^4} + 7\sqrt{1-x^4}}$$

for $x \in (0,1)$ follows from (51) and (55).

Case 3.2 $p = 1/\kappa$. Then Lemma 2.2 (2) and (54) lead to the conclusion that there exists $\xi^* \in (0,1)$ such that $H_p(x)$ is strictly increasing on $(0, \xi^*]$ and strictly decreasing on $[\xi^*, 1)$. Note that (53) becomes

$$\lim_{x \to 1^-} H_p(x) = 0. \tag{56}$$

It follows from (52) and (56) together with the piecewise monotonicity of $H_p(x)$ that

$$H_p(x) > 0 \tag{57}$$

for all $x \in (0,1)$.

Therefore,

$$arctlh(x) > \frac{5\kappa x}{5 + (7\kappa - 10)\sqrt[4]{1-x^4} + (5 - 2\kappa)\sqrt{1-x^4}}$$

for all $x \in (0,1)$ follows from (51) and (57).

Case 3.3 $37/75 < p < 1/\kappa$. Then it follows from (3.19), by Taylor expansion,

$$\lim_{x\to 0^+}\frac{H_p(x)}{x^9}=\frac{1}{16}\left(p-\frac{37}{75}\right)>0, \tag{58}$$

$$\lim_{x\to 1^-}H_p(x)<0. \tag{59}$$

Equations (51) and inequalities (58) and (59) imply that there exists small enough $0<\delta_3<1$ and large enough $0<\delta_3'<1$ such that

$$arctlh(x)>\frac{5x}{5p+(7-10p)\sqrt[4]{1-x^4}+(5p-2)\sqrt{1-x^4}}$$

for all $x\in(0,\delta_3)$ and

$$arctlh(x)<\frac{5x}{5p+(7-10p)\sqrt[4]{1-x^4}+(5p-2)\sqrt{1-x^4}}$$

for all $x\in(1-\delta_3',1)$.

Therefore, Theorem 3.3 follows from Cases 3.1- 3.3 and the monotonicity of $H_p(x)$.

Theorem 3.4 The double inequality

$$\frac{1}{\lambda_4+(7/5-2\lambda_4)\sqrt[4]{1+x^4}+(\lambda_4-2/5)\sqrt{1+x^4}}<\frac{arctl(x)}{x}$$

$$<\frac{1}{\mu_4+(7/5-2\mu_4)\sqrt[4]{1+x^4}+(\mu_4-2/5)\sqrt{1+x^4}}$$

holds for all $0<|x|<1$ if and only if $\lambda_4\geq 37/75=0.4933\ldots$ and $\mu_4\leq p_1=0.4857\ldots$, where p_1 is defined as in Lemma 2.2.

Proof. Without loss of generality, we may assume $x\in(0,1)$. Let $p\in(0,1)$ and

$$J_p(x)=arctl(x)-\frac{x}{p+(7/5-2p)\sqrt[4]{1+x^4}+(p-2/5)\sqrt{1+x^4}}. \tag{60}$$

Then simple computations lead to

$$\lim_{x\to 0^+}J_p(x)=0, \tag{61}$$

$$\lim_{x\to 1^-}J_p(x)=\tau-\frac{5}{5\left(1-\sqrt[4]{2}\right)^2p+\sqrt[4]{2}\left(7-2\sqrt[4]{2}\right)}, \tag{62}$$

$$J_p'(x)=\frac{\left(\sqrt[4]{1+x^4}-1\right)^2 g\left(\sqrt[4]{1+x^4}\right)}{\left(1+x^4\right)^{3/4}\left(5p+(7-10p)\sqrt[4]{1+x^4}+(5p-2)\sqrt{1+x^4}\right)^2} \tag{63}$$

where $g(\bullet)$ is defined as in Lemma 2.2.

We divide the proof into three cases.

Case 4.1 $p=37/75$. Then from Lemma 2.2 (1), (61) and (63) we clearly see that

$$J_p(x)>0 \tag{64}$$

for all $x\in(0,1)$.

Therefore,

$$arctl(x)>\frac{75x}{37+31\sqrt[4]{1+x^4}+7\sqrt{1+x^4}}$$

for $x\in(0,1)$ follows from (60) and (64).

Case 4.2 $p=p_1$. Then Lemma 2.2 (3) and (63) lead to the conclusion that there exists $\zeta^*\in(0,1)$ such that $J_p(x)$ is strictly decreasing on $(0,\zeta^*]$ and strictly increasing on $[\zeta^*,1)$. Note that (62) becomes

$$\lim_{x\to 1^-} J_p(x) = 0, \tag{65}$$

It follows from (61) and (65) together with the piecewise monotonicity of $J_p(x)$ that

$$J_p(x) < 0 \tag{66}$$

for all $x \in (0,1)$.

Therefore, the inequality

$$arctl(x) < \frac{5x}{5p_1 + (7-10p_1)\sqrt[4]{1+x^4} + (5p_1 - 2)\sqrt{1+x^4}}$$

for all $x \in (0,1)$ follows from (60) and (66).

Case 4.3 $p_1 < p < 37/75$. Then it follows from (60) and (3.30) that

$$\lim_{x\to 0^+} \frac{J_p(x)}{x^9} = \frac{1}{16}\left(p - \frac{37}{75}\right) < 0, \tag{67}$$

$$\lim_{x\to 1^-} J_p(x) > 0. \tag{68}$$

Equations (60) and inequalities (67) and (68) imply that there exists small enough $0 < \delta_4 < 1$ and large enough $0 < \delta_4' < 1$ such that

$$arctl(x) < \frac{5x}{5p + (7-10p)\sqrt[4]{1+x^4} + (5p - 2)\sqrt{1+x^4}}$$

for all $x \in (0, \delta_4)$ and

$$arctl(x) > \frac{5x}{5p + (7-10p)\sqrt[4]{1+x^4} + (5p - 2)\sqrt{1+x^4}}$$

for all $x \in (1-\delta_4', 1)$.

Therefore, Theorem 3.4 follows from Cases 4.1- 4.3 and the monotonicity of $J_p(x)$.

From Theorems 3.1- 3.4 we get Corollary 4.1 immediately.

Corollary 4.1 Let

$p_0 = \left[5 - \sqrt[4]{2}\left(8 - 3\sqrt[4]{2}\right)\sigma\right] / \left[5\left(1-\sqrt[4]{2}\right)^2 \sigma\right] = 0.6844\ldots$ and $p_1 = \left[5 - \sqrt[4]{2}\left(7 - 2\sqrt[4]{2}\right)\tau\right] / \left[5\left(1-\sqrt[4]{2}\right)^2 \tau\right] = 0.4857\ldots$, then the double inequalities

$$\frac{5\omega x}{5 + 2(4\omega - 5)\sqrt[4]{1-x^4} + (5-3\omega)\sqrt{1-x^4}} < arcsl(x) < \frac{75x}{52 + 16\sqrt[4]{1-x^4} + 7\sqrt{1-x^4}},$$

$$\frac{5x}{52 + 16\sqrt[4]{1+x^4} + 7\sqrt{1+x^4}} < arcslh(x) < \frac{5x}{5p_0 + 2(4-5p_0)\sqrt[4]{1+x^4} + (5p_0 - 3)\sqrt{1+x^4}},$$

$$\frac{5\kappa x}{5 + (7\kappa - 10)\sqrt[4]{1-x^4} + (5-2\kappa)\sqrt{1-x^4}} < arctlh(x) < \frac{75x}{37 + 31\sqrt[4]{1-x^4} + 7\sqrt{1-x^4}},$$

$$\frac{75x}{37+31\sqrt[4]{1+x^4}+7\sqrt{1+x^4}} < arctl(x) < \frac{5x}{5p_1+(7-10p_1)\sqrt[4]{1+x^4}+(5p_1-2)\sqrt{1+x^4}}$$

for all $0<|x|<1$.

References

[1] B. C. Carlson, Special functions of applied mathematics, Academic Press, New York (1977). https://doi.org/10.1137/1021080

[2] J. M. Borwein, P. B. Borwein, Pi and the AGM: A study in the analytic number theory and computational complexity, Wiley, New York (1998).

[3] B. C. Carlson, Algorithms involving arithmetic and geometric means, Amer. Math. Mon., 78 (1971), 496-505. https://doi.org/10.1080/00029890.1971.11992791

[4] E. Neuman, On Gauss lemniscate functions and lemniscatic mean, Math. Pannon., 18(1) (2007), 77-94.

[5] E. Neuman, Two-sided inequalities for the lemniscate functions, Journal of Inequalities and Special Functions, 1(2) (2010), 1-7.

[6] E. Neuman, On Gauss lemniscate functions and lemniscatic mean II, Math. Pannon., 23 (2012), 65- 73.

[7] F. W. J. Olver, D, W. Lozier, R. F. Boisvert and C. W. Clark, the NIST Handbook of Mathematical Functions, Cambridge University Press, New York (2010).

[8] B. N. Guo and F. Qi, Monotonicity of functions connected with the gamma function and the volume of the unit ball, Integral Transforms Spec. Funct., 23 (9) (2012), 701–708. https://doi.org/10.1080/10652469.2011.627511

[9] F. Qi, Bounds for the ratio of two gamma functions, J. Inequal. Appl., 2010(2010), Article ID 493058, 84 pages. https://doi.org/10.1155/2010/493058

[10] T.-H. Zhao, Y.-M. Chu, H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011(2011), Article ID 896483, 13 pages. https://doi.org/10.1155/2011/896483

[11] Y. M. Chu, S. L. Qiu and M. K. Wang, Sharp inequalities involving the power mean and complete elliptic integral of the first kind, Rocky Mountain J. Math., 43(5) (2013), 1489-1496. https://doi.org/10.1216/rmj-2013-43-5-1489

[12] Z. H. Yang, W.-M. Qian and Y. M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 21(4) (2018), 1185-1199. https://doi.org/10.7153/mia-2018-21-82

[13] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017 (2017), Article 210, 17 pages. https://doi.org/10.1186/s13660-017-1484-y

[14] T.-H. Zhao, M.-K. Wang and Y.-M. Chu, A sharp double inequality involving generalized complete elliptic integral of the first kind, AIMS Math., 5(5) (2020), 4512-4528. https://doi.org/10.3934/math.2020290

[15] W.-M. Qian, Z.-Y. He, Y.-M. Chu, Approximation for the complete elliptic integral of the first kind, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 114(2): 57, (2020), 1-12. https://doi.org/10.1007/s13398-020-00784-9

[16] C.-P. Chen, Wilker and Huygens type inequalities for the lemniscate functions, J. Math. Inequal., 6(4) (2012), 673- 684. https://doi.org/10.7153/jmi-06-65

[17] C.-P. Chen, Wilker and Huygens type inequalities for the lemniscate functionsII [J], Math. Inequal. Appl., 16 (2) (2013), 577-586. https://doi.org/10.7153/mia-16-43

[18] J.-E. Deng and C.-P. Chen, Sharp Shafer-Fink type inequalities for Gauss lemniscate functions, J. Inequal. Appl., 2014 (2014), Paper No. 35, 14 pages. https://doi.org/10.1186/1029-242x-2014-35

[19] J. Liu and C.-P. Chen, Padé approximant related to inequalities for Gauss lemniscate functions, J. Inequal. Appl., 2016, 2016: Art 320, http://dx.doi.org/10.1186/s13660-016-1262-2

[20] Ryo Nishimura, New properties of the lemniscate function and its transformation, J. Math. Anal. Appl., 427, 2015, 460-468. https://doi.org/10.1016/j.jmaa.2015.02.066

[21] M.-J. Wei, Y. He and G.-D. Wang, Shafer-Fink type inequalities for arc lemniscate functions, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 2020, 114(2): 53, 1-14. https://doi.org/10.1007/s13398-020-00782-x