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Improvements of Inequalities for Sándor and Identric Means with Applications

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Abstract: This paper is devoted to establishing the best possible upper and lower bounds for the Sándor mean and the identric mean in terms of the harmonic and arithmetic means. These two means, which are closely related to other classical means such as the logarithmic and Seiffert means, have attracted considerable attention in recent studies due to their rich mathematical structure and applications. The primary objective of this article is to refine existing inequalities by determining the sharpest constants in certain double inequalities involving convex combinations of the harmonic and arithmetic means and weighted geometric expressions. To achieve this, we apply a combination of classical inequality techniques and rigorous analysis of the monotonicity properties of auxiliary functions. Specifically, we construct appropriate comparison functions and examine their first and second derivatives to establish strict monotonicity and convexity. These analytic tools allow us to precisely identify the optimal parameters in the refined bounds for the Sándor and identric means. In particular, we show that the best possible bounds for these means can be expressed as weighted combinations of the expressions involving convex combinations of the harmonic and arithmetic means and weighted geometric expressions, where the weights are explicitly determined through monotonicity arguments. As an application of the main results, we derive new and sharp bounds for the functions of the inverse sine and the inverse hyperbolic tangent, which are important in various areas of mathematical analysis and approximation theory. Our findings improve upon several known results in the literature and provide a unified framework for analyzing the relationships among classical and non-classical means.

Keywords: Sándor Mean; Identric Mean; Harmonic Mean; Arithmetic Mean; Inequality

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1 Introduction

Let $p \in \mathbb{R}$, $u, v > 0$ with $u \neq v$. Then the harmonic mean $H(u,v)$, geometric mean $G(u,v)$, arithmetic mean $A(u,v)$, logarithmic mean $L(u,v)$, first Seiffert mean $P(u,v)$, Sándor mean $X(u,v)$, identric mean $I(u,v)$ and $p-$th power mean $M_p(u,v)$ are respectively defined by [1, 2].

$$H(u,v) = 2uv/\left(u+v\right), \quad G(u,v) = \sqrt{uv},$$

$$A(u,v) = \frac{u+v}{2}, \quad L(u,v) = \frac{u-v}{\log u - \log v},$$

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$$P(u,v)=\frac{u-v}{2\arcsin\left[(u-v)/(u+v)\right]}, \tag{1}$$

$$X(u,v)=A(u,v)e^{\frac{G(u,v)}{P(u,v)}-1}, \tag{2}$$

$$I(u,v)=\frac{1}{e}\left(v^v/u^u\right)^{1/(v-u)}=G(u,v)e^{\frac{A(u,v)}{L(u,v)}-1} \tag{3}$$

and

$$M_p(u,v)=\left[\left(u^p+v^p\right)/2\right]^{1/p}\ \left(p\neq 0\right),\ M_0(u,v)=\sqrt{uv}.$$

It is well-known that the power mean $M_p(u,v)$ is strictly increasing with respect to $p\in\mathbb{R}$ for $u,v>0$ with $u\neq v$, and the inequalities.

$$H(u,v)=M_{-1}(u,v)<G(u,v)=M_0(u,v)<L(u,v)<X(u,v)$$

$$<P(u,v)<I(u,v)<A(u,v)=M_1(u,v)$$

hold for all $u,v>0$ with $u\neq v$.

Recently, these means have been the subject of intensive research.

The authors proved that the double inequalities.

$$M_{\lambda_1}(u,v)<L(u,v)<M_{\mu_1}(u,v),\ M_{\lambda_2}(u,v)<X(u,v)<M_{\mu_2}(u,v),$$

$$M_{\lambda_3}(u,v)<P(u,v)<M_{\mu_3}(u,v),\ M_{\lambda_4}(u,v)<I(u,v)<M_{\mu_4}(u,v) \tag{4}$$

hold for all $u,v>0$ with $u\neq v$ if and only if $\lambda_1\leq 0$, $\mu_1\geq 1/3$, $\lambda_2\leq 1/3$, $\mu_2\geq \log 2/\left(1+\log 2\right)$, $\lambda_3\leq 1/2$, $\mu_3\geq 2/3$, $\lambda_4\leq 2/3$ and $\mu_4\geq \log 2$. (see [3-8])

The authors present bounds for $L(u,v)$ in terms of $G(u,v)$ and $A(u,v)$:

$$G^{2/3}(u,v)A^{1/3}(u,v)<L(u,v)<\frac{2}{3}G(u,v)+\frac{1}{3}A(u,v) \tag{5}$$

hold for all $u,v>0$ with $u\neq v$. (see [9-11])

Sándor established the inequalities

$$A^{2/3}(u,v)G^{1/3}(u,v)<P(u,v)<I(u,v) \tag{6}$$

hold for all $u,v>0$ with $u\neq v$.(see [12])

Qian et al., Xia and Chu proved that the double inequalities.

$$\alpha_1A(u,v)+\left(1-\alpha_1\right)H(u,v)<X(u,v)<\beta_1A(u,v)+\left(1-\beta_1\right)H(u,v), \tag{7}$$

$$\alpha_2A(u,v)+\left(1-\alpha_2\right)H(u,v)<I(u,v)<\beta_2A(u,v)+\left(1-\beta_2\right)H(u,v) \tag{8}$$

hold for all $u,v>0$ with $u\neq v$ if and only if $\alpha_1\leq 1/e$, $\beta_1\geq 2/3$, $\alpha_2\leq 2/e$ and $\beta_2\geq 5/6$.(see [13, 14])

Inequalities (4)-(8) and the identity $G(u,v)=\sqrt{A(u,v)H(u,v)}$ lead to

$$H^{1/3}(u,v)A^{2/3}(u,v)<X(u,v)<\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v), \tag{9}$$

$$H^{1/6}(u,v)A^{5/6}(u,v)<I(u,v)<\frac{1}{6}H(u,v)+\frac{5}{6}A(u,v) \tag{10}$$

hold for all $u,v>0$ with $u\neq v$.

More remarkable inequalities for Sándor mean or identric mean can be found in the literature [15-25].

Motivated by inequalities (9) and (10), it makes senses to ask what the optimal parameters $\alpha_1,\beta_1,\alpha_2,\beta_2\in(0,\ 1)$ satisfying

$$\alpha_1\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+(1-\alpha_1)H^{1/3}(u,v)A^{2/3}(u,v)<X(u,v)<\beta_1\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+(1-\beta_1)H^{1/3}(u,v)A^{2/3}(u,v)$$

$$\alpha_2\left[\frac{1}{6}H(u,v)+\frac{5}{6}A(u,v)\right]+(1-\alpha_2)H^{1/6}(u,v)A^{5/6}(u,v)<I(u,v)<\beta_2\left[\frac{1}{6}H(u,v)+\frac{5}{6}A(u,v)\right]+(1-\beta_2)H^{1/6}(u,v)A^{5/6}(u,v)$$

for all $u,v>0$ with $u\neq v$. The aim of article is to answer this question.

2 Lemmas

In order to establish our main results we need two lemmas, which we present in this section.

Lemma 2.1. Let $p\in(0,1)$ and

$$f(t)=3p^2t^{10}+14p(1-p)t^6+18p^2t^4-9(1-p)^2t^2-2p(1-p) \tag{11}$$

Then the following statements are true:

If $p=3/10$, then $f(t)<0$ for all $t\in(0,1)$;

(2) If $p=3/(2e)=0.5518\cdots$, then there exists $\lambda\in(0,1)$ such that $f(t)<0$ for $t\in(0,\lambda)$ and $f(t)>0$ for $t\in(\lambda,1)$.

Proof (1) If $p=3/10$, then (11) reduces to

$$f(t)=-\frac{3}{100}(1-t^2)(9t^8+9t^6+107t^4+161t^2+14)<0 \tag{12}$$

for $t\in(0,1)$.

Therefore, part (1) follows easily from (12).

(2) If $p=3/(2e)$, Let $f_1(t)=f'(t)/(6t)$, then straightforward computations yield

$$f(0)=-2p(1-p)<0,\ f(1)=30\left(p-\frac{3}{10}\right)>0, \tag{13}$$

$$f_1(t)=5p^2t^8+14p(1-p)t^4+12p^2t^2-3(1-p)^2,$$

$$f_1(0)=-3(1-p)^2<0,\ f_1(1)=20\left(p-\frac{3}{20}\right)>0. \tag{14}$$

$$f_1'(t)=8pt\left[5pt^6+7(1-p)t^2+3p\right]>0 \tag{15}$$

for $t\in(0,1)$.

From (15) we clearly see that $f_1(t)$ is strictly increasing on $(0,1)$. Then (14) lead to the conclusion that there exists $\lambda_0\in(0,1)$ such that $f(t)$ is strictly decreasing on $(0,\lambda_0]$ and strictly decreasing on $[\lambda_0,1)$.

Therefore, part (2) follows from (13) and the piecewise monotonicity of $f(t)$.

Lemma 2.2. Let $q\in(0,1)$ and

$$g(t)=3q^2t^{11}+56q(1-q)t^6+75q^2t^5-72(1-q)^2t-50q(1-q) \tag{16}$$

Then the following statements are true:

(1) If $q=12/25$, then $g(t)<0$ for all $t\in(0,1)$;

(2) If $q=12/(5e)=0.8829\cdots$, then there exists $\mu\in(0,1)$ such that $g(t)<0$ for $t\in(0,\mu)$ and $g(t)>0$ for $t\in(\mu,1)$.

Proof (1) If $q=12/25$, then (16) becomes

$$g(t)=-\frac{24}{625}(1-t)\left(18t^{10}+18t^{9}+18t^{8}+18t^{7}+18t^{6}+382t^{5}+832t^{4}+832t^{3}+832t^{2}+832t+325\right)<0 \quad (17)$$

for $t\in(0,1)$.

Therefore, part (1) follows easily from (17).

(2) If $q=12/(5e)$, then simple computations lead to

$$g(0)=-50q(1-q)<0,\ g_1(1)=150\left(q-\frac{12}{25}\right)>0, \quad (18)$$

$$g'(t)=3\left[11q^2t^{10}+112q(1-q)t^5+125q^2t^4-24(1-q)^2\right]$$

$$g'(0)=-72(1-q)^2<0,\ g'(1)=480\left(q-\frac{3}{20}\right)>0, \quad (19)$$

$$g''(t)=30qt^3\left[11qt^6+56(1-q)t+50q\right]>0 \quad (20)$$

for $x\in(0,1)$.

It follows from (20) we clearly see that $g'(t)$ is strictly increasing on $(0,1)$. Then (19) imply that there exists $\mu_0\in(0,1)$ such that $g(t)$ is strictly decreasing on $(0,\mu_0]$ and strictly increasing on $[\mu_0,1)$.

Therefore, part (2) follows from (18) and the piecewise monotonicity of $g(t)$.

3 Main Results

Theorem 3.1 The double inequality

$$\alpha_1\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+(1-\alpha_1)H^{1/3}(u,v)A^{2/3}(u,v)<X(u,v)<\beta_1\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+(1-\beta_1)H^{1/3}(u,v)A^{2/3}(u,v)$$

holds for all $u,v>0$ with $u\neq v$ if and only if $\alpha_1\leq 3/10$ and $\beta_1\geq 3/(2e)=0.5518\cdots$.

Proof. Since $H(u,v)$, $X(u,v)$ and $A(u,v)$ are symmetric and homogenous of degree one, without loss of generality, we assume that $u>v>0$. Let $p\in(0,1)$ and $x=(u-v)/(u+v)\in(0,1)$. Then (1) and (2) lead to

$$X(u,v)=A(u,v)e^{\sqrt{1-x^2}\arcsin(x)/x-1},\ H(u,v)=A(u,v)(1-x^2),$$

$$\log X(u,v)-\log p\left[H(u,v)/3+2A(u,v)/3\right]+(1-p)H^{1/3}(u,v)A^{2/3}(u,v)$$

$$=\frac{\sqrt{1-x^2}\arcsin(x)}{x}-\log\left[p(1-x^2/3)+(1-p)(1-x^2)^{1/3}\right]-1. \quad (21)$$

Let

$$F(x)=\frac{\sqrt{1-x^2}\arcsin(x)}{x}-\log\left[p\left(1-x^2/3\right)+(1-p)\left(1-x^2\right)^{1/3}\right]-1. \tag{22}$$

Then elaborated computations lead to

$$F\left(0^+\right)=0, \tag{23}$$

$$F\left(1^-\right)=\log 3-\log(2p)-1, \tag{24}$$

$$F'(x)=\frac{1}{x^2\sqrt{1-x^2}}F_1(x), \tag{25}$$

where

$$F_1(x)=\frac{x\left[p\left(3+x^2\right)\left(1-x^2\right)^{2/3}+(1-p)\left(3-x^2\right)\right]}{\left(1-x^2\right)^{1/6}\left[p\left(3-x^2\right)1+3(1-p)\left(1-x^2\right)^{1/3}\right]}-\arcsin(x)$$

Moreover,

$$F_1\left(0^+\right)=0,\quad F_1\left(1^-\right)=\infty, \tag{26}$$

$$F_1'(x)=-\frac{6x^4}{\left(1-x^2\right)^{7/6}\left[p\left(3-x^2\right)+3(1-p)\left(1-x^2\right)^{1/3}\right]^2}f\left(\sqrt[6]{1-x^2}\right) \tag{27}$$

where $f(\bullet)$ is defined as in Lemma 2.1.

We divide the proof into four cases.

Case 1: $p=3/10$. Then (21)-(23) and (25)-(27) together with Lemma 2.1 (1) that

$$X(u,v)>\frac{3}{10}\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+\frac{7}{10}H^{1/3}(u,v)A^{2/3}(u,v)$$

holds for all $u,v>0$ with $u\neq v$.

Case 2: $3/10<p<1$. Let $x>0$ and $x\to 0^+$. Then power series expansions lead to

$$F(x)=\frac{1}{9}\left(\frac{3}{10}-p\right)x^4+o\left(x^6\right). \tag{28}$$

Equations (21), (22) and (28) lead to the conclusion that there exists small enough $0<\delta_1<1$ such that

$$X(u,v)<p\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+(1-p)H^{1/3}(u,v)A^{2/3}(u,v)$$

for all $u,v>0$ with $(u-v)/(u+v)\in(0,\delta_1)$.

Case 3: $p=3/(2e)$. Then (27) and Lemma 2.1 (2) lead to the conclusion that there exists $\lambda_1\in(0,1)$ such that $F_1(x)$ is strictly decreasing on $(0,\lambda_1)$ and strictly increasing on $(\lambda_1,1)$. Then from (25) and (26), we know that there exists $\lambda_1^*\in(0,1)$ such that $F(x)$ is strictly decreasing on $(0,\lambda_1^*)$ and strictly increasing on $(\lambda_1^*,1)$. Note that (24) becomes

$$F\left(1^{-}\right)=0. \tag{29}$$

Therefore,

$$X\left(u,v\right)<\frac{3}{2e}\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+\left(1-\frac{3}{2e}\right)H^{1/3}(u,v)A^{2/3}(u,v)$$

follows from (21)-(23) and (29) together with the piecewise monotonicity of $F\left(x\right)$.

Case 4: $0<p<3/\left(2e\right)$. Then (24) leads to

$$F\left(1^{-}\right)=\log 3-\log\left(2p\right)-1>0. \tag{30}$$

Equations (21), (22) and inequality (30) imply that there exists large enough $0<\delta_1^*<1$ such that

$$X\left(u,v\right)>p\left[\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right]+\left(1-p\right)H^{1/3}(u,v)A^{2/3}(u,v)$$

for all $u>v>0$ with $\left(u-v\right)/\left(u+v\right)\in\left(1-\delta_1^*,1\right)$.

Theorem 3.2 The double inequality

$$\alpha_2\left[\frac{1}{6}H(u,v)+\frac{5}{6}A(u,v)\right]+\left(1-\alpha_2\right)H^{1/6}(u,v)A^{5/6}(u,v)<I(u,v)<\beta_2\left[\frac{1}{6}H(u,v)+\frac{5}{6}A(u,v)\right]+\left(1-\beta_2\right)H^{1/6}(u,v)A^{5/6}(u,v)$$

holds for all $u,v>0$ with $u\neq v$ if and only if $\alpha_2\leq 12/25$ and $\beta_2\geq 12/\left(5\mathrm{e}\right)=0.8829\cdots$.

Proof. Since $H(u,v)$, $I(u,v)$ and $A(u,v)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $u>v>0$. Let $q\in\left(0,1\right)$ and $x=\left(u-v\right)/\left(u+v\right)\in\left(0,1\right)$. Then (1), (3) lead to

$$I\left(u,v\right)=A\left(u,v\right)\sqrt{1-x^2}\,e^{\frac{\arctan h(x)}{x}-1},\ H(u,v)=A(u,v)\left(1-x^2\right),$$

$$\log I(u,v)-\log q\left[H(u,v)/6+5A(u,v)/6\right]+\left(1-q\right)H^{1/6}(u,v)A^{5/6}(u,v)$$

$$=\log\left(\sqrt{1-x^2}\right)+\frac{\arctan h\left(x\right)}{x}-\log\left[q\left(1-\frac{1}{6}x^2\right)+\left(1-q\right)\left(1-x^2\right)^{1/6}\right]-1 \tag{31}$$

Let

$$G\left(x\right)=\log\left(\sqrt{1-x^2}\right)+\frac{\arctan h\left(x\right)}{x}-\log\left[q\left(1-\frac{1}{6}x^2\right)+\left(1-q\right)\left(1-x^2\right)^{1/6}\right]-1. \tag{32}$$

Then elaborated computations lead to

$$G\left(0^{+}\right)=0, \tag{33}$$

$$G\left(1^{-}\right)=\log 12-\log\left(5q\right)-1, \tag{34}$$

$$G'\left(x\right)=\frac{G_1\left(x\right)}{x^2}, \tag{35}$$

where

$$G_1(x)=\frac{x\left[2(1-q)\left(3-x^2\right)+q\left(6+x^2\right)\left(1-x^2\right)^{5/6}\right]}{\left(1-x^2\right)^{5/6}\left[q\left(6-x^2\right)+6(1-q)\left(1-x^2\right)^{1/6}\right]}-\arctan h(x),$$

$$G_1\left(0^+\right)=0,\ G_1\left(1^-\right)=\infty, \tag{36}$$

$$G_1'(x)=-\frac{x^4 g\left(\sqrt[6]{1-x^2}\right)}{3\left(1-x^2\right)^{11/6}\left[q\left(6-x^2\right)+6(1-q)(1-x)^{1/6}\right]^2} \tag{37}$$

where $g(\bullet)$ is defined as in Lemma 2.2.

We divide the proof into four cases.

Case 1: $q=12/25$. Then (31)-(33) and (35)-(37) together with Lemma 2.2 (1) lead to the conclusion that

$$I(u,v)>\frac{12}{25}\left[\frac{5}{6}A(u,v)+\frac{1}{6}H(u,v)\right]+\frac{13}{25}A^{5/6}(u,v)H^{1/6}(u,v)$$

for all $u,v>0$ with $u\neq v$.

Case 2: $12/25<q<1$. Let $x>0$ and $x\to 0^+$. Then power series expansions lead to

$$G(x)=\frac{5}{72}\left(\frac{12}{25}-q\right)x^4+O\left(x^6\right). \tag{38}$$

Equations (31), (32), and (38) imply that there exists small enough $0<\delta_2<1$ such that

$$I(u,v)<q\left[\frac{5}{6}A(u,v)+\frac{1}{6}H(u,v)\right]+(1-q)A^{5/6}(u,v)H^{1/6}(u,v)$$

for all $u,v>0$ with $(u-v)/(u+v)\in(0,\delta_2)$.

Case 3: $q=12/(5e)$. Then (37) and Lemma 2.2 (2) lead to the conclusion that there exists.

$\mu_1\in(0,1)$ such that $G_1(x)$ is strictly decreasing on $(0,\mu_1)$ and strictly increasing on $(\mu_1,1)$. Then from (35) and (36), we clearly see that there exists $\mu_1^*\in(0,1)$ such that $G(x)$ is strictly decreasing on $(0,\mu_1^*)$ and strictly increasing on $(\mu_1^*,1)$. Note that (34) leads to

$$G\left(1^-\right)=0. \tag{39}$$

Therefore,

$$I(u,v)<\frac{12}{5e}\left[\frac{5}{6}A(u,v)+\frac{1}{6}H(u,v)\right]+\left(1-\frac{12}{5e}\right)A^{5/6}(u,v)H^{1/6}(u,v)$$

follows from (31)-(33) and (39) together with the piecewise monotonicity of $G(x)$.

Case 4: $0<q<12/(5e)$. Then from (34) we get

$$G\left(1^-\right)=\log 12-\log(5q)-1>0. \tag{40}$$

Equations (31), (32) and inequality (40) imply that there exists large enough $0<\delta_2^*<1$ such that

$$I(u,v) > q\left[\frac{5}{6}A(u,v)+\frac{1}{6}H(u,v)\right]+(1-q)A^{5/6}(u,v)H^{1/6}(u,v)$$

for all $u>v>0$ with $(u-v)/(u+v)\in(1-\delta_2^*,1)$.

As an application, then from Theorems 3.1 and 3.2 we get the following Corollary 3.3 and 3.4 immediately.

Corollary 3.3 Let

$$\xi(u,v;\alpha)=\log\left[\alpha\left(\frac{1}{3}H(u,v)+\frac{2}{3}A(u,v)\right)+(1-\alpha)H^{1/3}(u,v)A^{2/3}(u,v)\right]-\log A(u,v)+1,$$

$$\zeta(u,v;\beta)=\log\left[\beta\left[\frac{1}{6}H(u,v)+\frac{5}{6}A(u,v)\right]+(1-\beta)H^{1/6}(u,v)A^{5/6}(u,v)\right]-\log G(u,v)+1.$$

The double inequalities

$$\frac{G(u,v)}{\xi(u,v;\beta_1)}<P(u,v)<\frac{G(u,v)}{\xi(u,v;\alpha_1)} \text{ and } \frac{A(u,v)}{\zeta(u,v;\alpha_2)}<L(u,v)<\frac{A(u,v)}{\zeta(u,v;\beta_2)}$$

hold for all $u,v>0$ with $u\neq v$ if and only if $\alpha_1\le 3/10$, $\beta_1\ge 3/(2e)=0.5518\cdots$, $\alpha_2\le 12/25$ and $\beta_2\ge 12/(5\mathrm{e})=0.8829\cdots$.

Corollary 3.4 The double inequalities

$$\frac{\log\left[e\left((3-x^2)+7(1-x^2)^{1/3}\right)/10\right]}{\sqrt{1-x^2}}<\frac{\arcsin(x)}{x}<\frac{\log\left[\left((3-x^2)+(2e-3)(1-x^2)^{1/3}\right)/2\right]}{\sqrt{1-x^2}},$$

$$\log\left[\frac{e\left(2(6-x^2)+13(1-x^2)^{1/6}\right)}{25\sqrt{1-x^2}}\right]<\frac{\operatorname{arctan}h(x)}{x}<\log\left[\frac{2(6-x^2)+(5e-12)(1-x^2)^{1/6}}{5\sqrt{1-x^2}}\right]$$

hold for all $x\in(0,1)$.

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